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BAO QI-SHOU 保其壽 AND HIS POLYHEDRAL HUN YUAN TU 渾圓圖*

I. INTRODUCTION

The construction of combinatorial mathematics commonly known as a 'magic square' is an arrangement of numbers 1, 2, 3, . . . , n^2 into a square array such that the sum of the numbers along any row, column, or main diagonal is a fixed constant. It is usually agreed that the following diagram

4	9	2
3	5	7
8	1	6

was the first magic square ever created in the world. Its origin is shrouded in the mystical legends of ancient China. It came to be known as 'Luo Shu' 洛書 (Luo river writing). There was no clear connection between this configuration and mathematical study until the time of Yang Hui 楊輝, even though Zhen Luan 甄鸞 (sixth century) described it in his commentary to the *Shu-Shu Ji-Yi* 《數術記遺》 (*Memoir on Some Traditions of Mathematical Art*) of c. 190. In the year 1275, Yang Hui published his *Xu Gu Zhai Qi Suan-Fa* 《續古摘奇算法》 (*Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers*). The first section of the first chapter lists twenty 'zong heng tu' 縱橫圖. (Literally, 'vertical and horizontal diagram'.) The term actually encompasses a wider class of configurations than does the term 'magic square'. Besides magic squares, there are other, circular, arrangements of consecutive numbers 1, 2, 3, . . . , n among these diagrams of 'zong heng tu'. However, in each case, a constant sum along any of the prescribed directions always emerges. Yang's book contains the earliest known reference to the term 'zong heng tu'.

Constructions of 'zong heng tu' were further pursued by mathematicians such as Din Yi-Dong 丁易東 in the *Da-Yan Suo-Yin* 《大衍索隱》 (*Elucidation of the Da-Yan Method*) of the Song Dynasty,

Cheng-hung Lin and Daiwie Fu (Eds.), Philosophy and Conceptual History of Science in Taiwan, 209–220.

Cheng Da-Wei 程大位 in the *Suan-Fa Tong-Zong*《算法統宗》(*Systematic Treatise on Arithmetic*) of 1593, Fang Zhong-Tong 方中通 in the *Shu Du Yan*《數度衍》(*Generalization on Numbers and Measures*) of 1661, and the more innovative Zhang Chao 張潮 in his *Xin-Zhai Za-Cu*《心齋雜俎》(*Miscellanea of Xin Zhai*) of 1670. Such activities came to an almost complete halt after the influential mathematician Mei Jue-Cheng 梅瑴成 removed all 'zong heng tu' from his 1760 edition of the popular *Suan-Fa Tong-Zong*. However, Cheng Da Wei's original work was transmitted to Japan and provided a great impetus to the Japanese mathematical development. The construction of magic squares and circles henceforth flourished in Japan in the seventeenth and eighteenth centuries.

When one attempts to make a circular 'zong heng tu,' the most elementary method which comes to mind is an arrangement of complementary number pairs, i.e., $(1, n)$, $(2, n - 1)$, $(3, n - 2)$, . . . , which obviously sum to the constant $n + 1$. From a mathematical point of view, configurations based on this method are of lesser interest. Most Chinese magic circles and notable Japanese ones by Isomura Kittoku and Seki Kowa were of this type.

II. BAO QI-SHOU AND HIS WORK

The construction of 'zong heng tu' was literally lifted to a higher dimension when Bao Qi-Shou 保其壽 finally appeared on the scene. Bao Qi-Shou lived in the latter part of the nineteenth century. All that is known about his life is documented in a 341-character biography in the mimeographed *Nantong Xian Zhi*《南通縣志》(*Illustrated History of Nantong County*) which is kept in the municipal library of Nantong City.

His parents died when he was young. He was not able to pursue the traditional career as a scholar-bureaucrat. He tried his hand at business without success. When the Boxer Rebellion started, he enlisted in the army, but he was assigned to a clerical position, which was not his inclination. Finally, he left the army and returned to his studies. His personality was not particularly pleasant, and he was fond of writing satirical poems. Astronomy and mathematics were his true love. He regretted that most people shunned Occidental knowledge. He, at one time, chose gifted pupils and taught them arithmetic and algebra. He died when he was sixty-four. The year of his death is not clear.

My interest in Bao's combinatorial constructions was first aroused in 1980 when I read Li Yan's 李儼 (1935) article 'Zhong-Suan-Jia Zhi Zong Heng Tu Yan-Jiu' 〈中算家之縱横圖研究〉('A Study of Chinese Mathematicians' Magic Squares'). According to Li's account, Bao exhibited his diagrams in an article 'Zen Bu Suan Fa Hun Yuan Tu' 〈增補算法渾圓圖〉('Augmented Solid and Spherical Mathematical Diagrams') included in his privately published book Bi-Nai-Shan Fang-Ji《碧奈山房集》(*Bi-Nai Mountain Hut Records*) of c. 1880. Although Li re-drew Bao's original diagrams and changed traditional numeral symbols into current numeral symbols, he gave the reader a strong impression that he had quoted Bao's entire article. It seems that later authoritative historians of Chinese mathematics, such as Lam (1977) and Needham (1959), made their comments about Bao's work based on Li's version. However, they did not fully explore the subtleties of Bao's diagrams and failed to give him proper recognition. Since the few libraries which have Bao's book in their collections are quite inaccessible, a copy of his 'hun yuan tu' article was forwarded to me for study only recently. I am not certain whether there are other parts of his book containing mathematical matters of relevance and merit. Nevertheless, Bao's own article reveals more than Li's version conveyed.

III. ZEN BU SUAN-FA HUN YUAN TU〈增補算法渾圓圖〉

There are two volumes to Bao's book. Figure 1 reproduces the first page of Volume Two, which is the volume containing his diagrams. The article 'Zen Bu Suan-Fa Hun Yuan Tu' occupies twenty-four pages. Since there are no page numbers in the original volume, my page count starts from the beginning of this article. I will also use 'Lx' to indicate that the same diagram appears as Figure x in Li Yan's aforementioned article. It is not feasible for me to reproduce all the diagrams in this paper. Therefore, references to Li Yan's more accessible book should be useful to the reader.

Pages 1 and 2 contain the foreword of the article. The translation is as follows.

There are 25 diagrams in the *Xin-Zhai Za-Cu*. Zhang Chao said that these were not included in the *Suan-Fa Tong-Zong*. They were derived along similar lines and many more could be constructed. I have not seen the *Suan Fa Tong Zong*. However, what Zhang Chao derived were plane diagrams. He did not realize that cubes and spheres could be of more interest. All of these diagrams have their origins in the 'Luo Shu'. The

Fig. 1.

ingenuity of their construction is unbelievable. Nature possesses such wonderful things and reveals them through Zhang Chao's hands and mine.

What Bao did was to put numeral labels 1, 2, 3, . . . up to a suitable number on the vertices, edges, and faces of tetrahedra, cubes, octahedra, dodecahedra, icosahedra, and icosidodecahedra (a 32 sided figure with 12 pentagonal faces and 20 triangular faces) so that the sum of the vertex, edge, and face labels on a face of a given number of sides is a constant. This means that all face sums are equal except on the icosidodecahedron, which has triangular and pentagonal faces, hence two kinds of face sums. The constant-sum phenomenon preserves the essence of 'zong heng tu'. However, Bao called them 'hun yuan tu' 渾圓圖 to emphasize their three dimensionality. Below are some comments on Bao's constructions.

IV. THE GROUP OF CUBES

Page 4 (L51). This is essentially an arrangement of complementary pairs (1, 8), (2, 7), (3, 6), (4, 5). Since labels are placed on vertices only, it is also called a vertex labeling.

Page 5 (L52). This is said to be an edge labeling since only edges are labeled.

Page 6 (L53). The label 2 on the top edge should be 20. Both Bao's original and Li's depiction are incorrect here. Li omitted Bao's statement that this construction is obtained by combining the previous two. From this statement we know that Bao was aware of the general method by which a new 'hun yuan tu' arises when a figure with only vertex labels is superimposed over an identical figure with only edge labels. In order to conform to the principle that the number labels cover all numbers from 1 to n without repeating, the numbers for either the edge or the vertex labels must all be increased by a constant amount equal to the last vertex or edge label, respectively.

Page 7 (L54). This labeling also appears in Needham (1959). It illustrates a second general principle that complementary pairs can be added to a vertex labeling to obtain a new 'hun yuan tu'.

V. THE GROUP OF TETRAHEDRA

Page 8 (L55). This is a straightforward labeling, by complementary pairs, of the edges of a tetrahedron viewed downward from the top vertex.

Page 9 (see Figure 2). Li Yan did not include this diagram in his article. Without labels, this triangular diagram is a so-called planar net representation of a tetrahedron, i.e., a tetrahedron will be obtained when the three exterior triangles on the rim are folded up and the three number 1 vertices are identified as one. Bao states that each triangular face needs its own label. This also indicates the following third general principle that Bao was aware of. A labeling is said to be of the consecutive type if the face sums form a consecutive sequence of numbers $m, m + 1, \ldots, m + k$. It is straightforward to convert a consecutive type labeling into a constant-sum labeling by filling in another sequence of suitable consecutive numbers as face labels. When the numbers 5, 6, 7, 8 are removed, Figure 2 becomes a consecutive type labeling of tetrahedron.

Page 10 (L56). This is the same labeling as Figure 2 but this figure is constructed in the form of a solid tetrahedron. The meaning of Bao's statement 'xin tong ma she pi qiu' 形同馬射皮毬 is unclear to me.

Page 11 (L57, see Figure 3). Bao states that a constant-sum labeling cannot be produced if 1, 2, 3, 4 are placed on vertices and 5 to 10 are placed on edges. He also states that the face sum is not divisible by 4 if

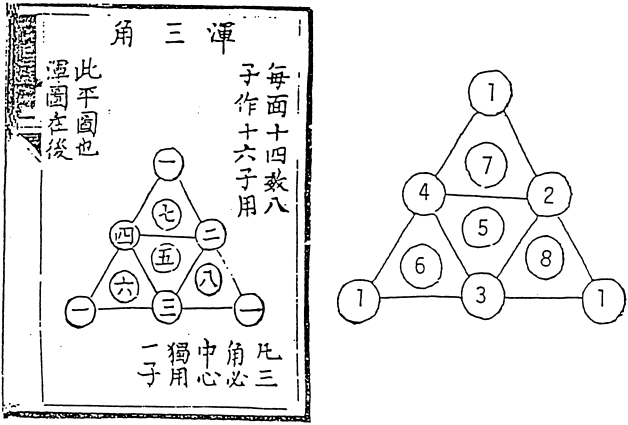

Fig. 2.

1 to 16 are used and 1, 2, 3, 4 are placed on vertices. Since each vertex of a tetrahedron is adjacent to all other vertices, there is essentially only one way to consecutively label the vertices, 1, 2, 3, 4. By trial and error, it is easy to see that there is no way to produce constant face sums with consecutively labeled edges. This verifies Bao's first statement. If the constant-sum phenomenon is going to occur, then the total sum must be divisible by 4 when each vertex label is counted three times and each edge label is counted twice. Now $(1 + 2 + 3 + 4) \times 3 + (5 + \ldots + 16) \times 2 = 282$ which is not a multiple of 4. So Bao's second claim makes sense.

Page 12 (L58, L59). A fourth general principle is expressed by Bao on this page. If a 'hun yuan tu' is labeled $1, 2, \ldots, n$, then a new 'hun yuan tu' results when each number k is switched to its complement $n + 1 - k$.

VI. THE GROUP OF OCTAHEDRA

Page 3 (L49, L50). Two representations of the same labeling, which is based on a consecutive vertex labeling, are exhibited here.

Page 13 (L60). The method of switching to complements is explicitly

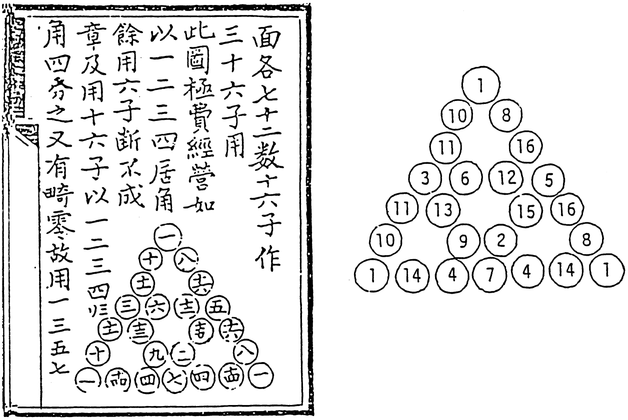

Fig. 3.

stated on this page. Another exchange of labels is also expressed here which does not appear in Li's quotation. The exchange is done by replacing 1 to 12 by 7 to 18 and 13 to 18 by 1 to 6, respectively.

Page 14 (L61). This is an edge labeling of complementary pairs. The correct face sums, when these pairs are added to the previous labeling and its complement, should be 231 and 249 instead of the stated 109 and 127.

VII. THE GROUP OF ICOSAHEDRA

Page 15 (L62, see Figure 4). The numbers 22 and 50 should be placed on the edge between the labels 9 and 15. Li's version has these numbers filled in. However, his 55 between 8 and 19 is 65 on the original and the positions of 42 and 45 are interchanged.

Page 16 (L63). The number 24 between 70 and 2 is a mistake. Li's version has it changed to 80.

VIII. THE GROUP OF DODECAHEDRA

Page 17 (L64). The vertices are consecutively labeled.

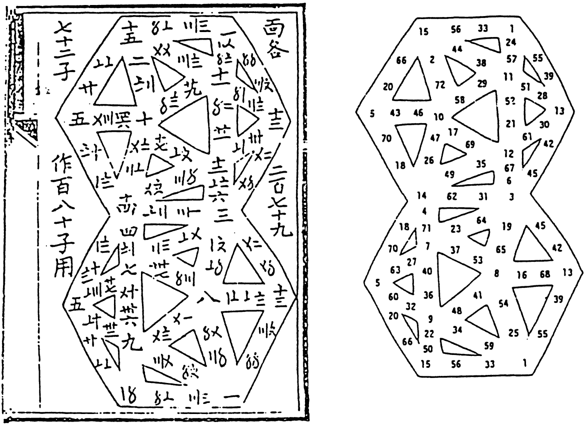

Fig. 4.

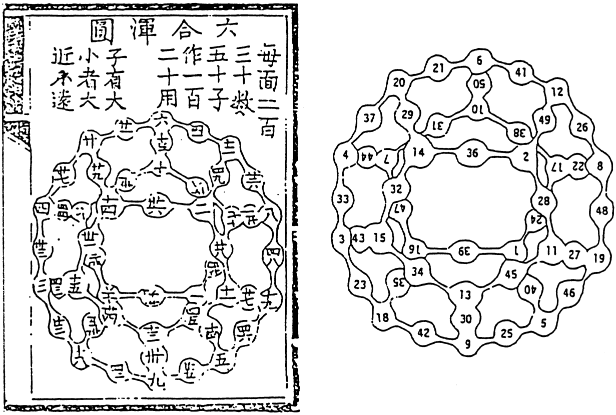

Fig. 5.

Page 19 (L65, see Figure 5). Li's version is drawn as a planar-net representation. Bao's original diagram looks more interesting in that the labels on the back are printed upside down. An amazing fact about this diagram is that the vertex labels are exactly the same as those on page 17. So when the vertex labels are removed and 20 is subtracted from each edge label, a figure whose edges are consecutively labeled is formed and is complementary to the figure whose vertices are consecutively labeled on page 17 to make a constant-sum labeling of both vertices and edges. This coupling method is a refined version of the third general principle used on page 9.

Page 18 (see Figure 6). This diagram is deleted in Li's version. However, it explicitly proves that a vertex labeling of the constant-sum type does not exist.

IX. THE GROUP OF ICOSIDODECAHEDRA

Page 23 (see Figure 7). The diagram shows a ball wrapped by six intertwining ribbons. Regarding the ribbons as edges, this configuration

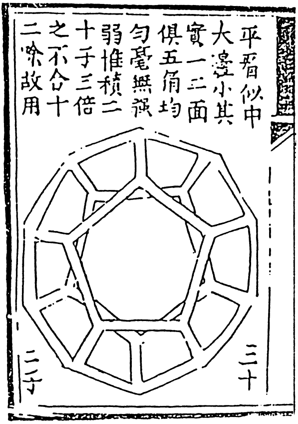

Fig. 6.

Fig. 7.

is combinatorially the same as an icosidodecahedron. There are two types of faces, namely triangles (20) and pentagons (12). Bao uses a divisibility proof to show that a vertex labeling of the constant-sum type does not exist.

Page 20 (L66, see Figure 8). This is a rather complicated construction. It would be tedious work to arrive at this labeling by trial and error. Whether Bao employed some systematic method is difficult to ascertain.

Page 21 (L67). The way to get this labeling is to add 60 to each vertex and subtract 30 from each edge in the previous diagram, then reflect the resulting labeling through a vertical mirror.

Page 22 (L68). This is complementary to the labeling of page 20. The number 36 should be added between 77 and 79 as in Li's version.

Page 24. This is a page of epilogue. It is translated as follows.

Further diagrams can be constructed by the previous methods. It becomes easier to make variations when more and more numbers are used. However, it is pointless to give those illustrations when all methods have been exhaustively demonstrated.

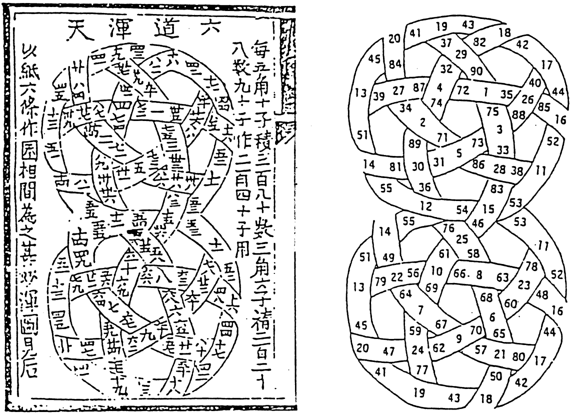

Fig. 8.

X. CONCLUDING REMARKS

The above examination of Bao's article clearly shows that he was consciously constructing labelings of polyhedra and that he was aware of at least four general methods:

(i) The merging of the constant-sum type.
(ii) The addition of complementary number pairs.
(iii) The merging of two complementary labelings of the consectuve type.
(iv) The switching of k with $n + 1 - k$.

Bao's achievement deserves to be ranked among the highest in combinatorial mathematics in the nineteenth century. It seems that no comparable combinatorial constructions of a similar kind existed then in the Western world even though, by that time, European mathematicians had advanced far beyond their Chinese colleagues on almost every other front.

Another point of importance in Bao's constructions comes from the possibility of further generalizations. Interpreted in the modern terminology of graph theory, Bao's constructions are labelings of plane graphs, i.e., graphs drawn on the Euclidean plane such that edges do not cross each other except at vertices. Both constant-sum and consecutive labelings can be defined for general plane graphs. This leads to interesting and challenging problems in graph theory. In an earlier paper (Lih, 1983), I have succeeded in making such constructions for several infinite families of graphs. A consecutive vertex labeling of the icosahedron is also included in that paper. This is one of the basic labelings of platonic polyhedra missed by Bao. However, I have to leave the existence of a consecutive edge labeling complementary to it as an open question.

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NOTE

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