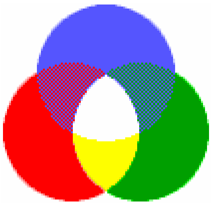
Fundamental Solution in Pandiagonal Magic Squares of Order Four and Five



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Pandiagonal Magic Squares of Order Four:

Before: 3 x 16 = 48
Now: 1 x 3 x 16 = 48

Pandiagonal Magic Squares of Order Five:

Before: 144 x 25 = 3600
Now: 1 x 9 x 16 x 25 = 3600

¡Each One of the Pandiagonal Magic Squares is Fundamental!

> *"What makes me tick is an aesthetic sense of order, of essential simplicity behind the apparent complexity. As an artist, it is possible to create exuberant and unique objects from a small and limited set of elements and rules; as a scientist, it is a challenge to discover a simple explanation for complex behavior, a general causal structure for a series of related but unique events. In this view, science and art are both aesthetic activities: only the direction of the approach differs."*
>
> Lionel March (1972)

2010

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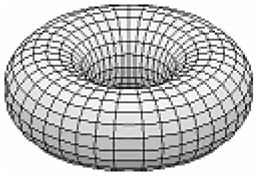

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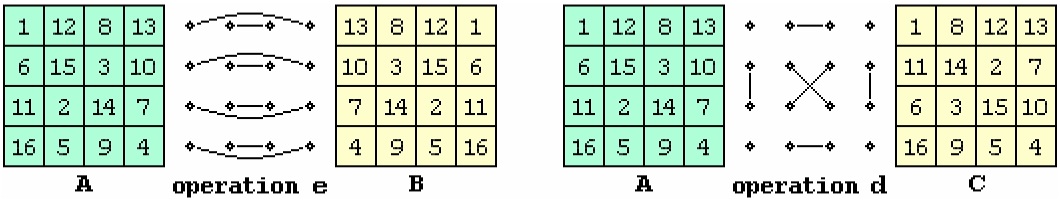
Introduction

In 1998, I learned the concept of *fundamental solutions* of magic squares in the web page of Mutsumi Suzuki about the 220 *fundamental solutions* of the 880 magic squares of order four; but I don't remember to have read a specific definition for that concept.
Applying a definition of compactness in magic squares of order five, the result was the well known 3600 pandiagonal magic squares and my investigation about the *symmetry operations* on this kind of magic squares; it returned me to the concept of *fundamental solutions* and the necessity of a definition.

Symmetry Operations on Magic Squares

A commutative operation on the entries of a magic square of order n it is a *symmetry operation* if the condition of magic square remains invariant; for example:



A

1	12	8	13
6	15	3	10
11	2	14	7
16	5	9	4

B

13	8	12	1
10	3	15	6
7	14	2	11
4	9	5	16

C

1	8	12	13
11	14	2	7
6	3	15	10
16	9	5	4

The squares **B** and **C** are magic squares ; **B** = **A** and **C** ≠ **A**

If the magic square has some additional condition, the commutative operation on the entries it is a *symmetry operation* if the additional condition also remains invariant. In this paper, are of interest the *symmetry operations* of the type **d** (the resulting square of the *symmetry operation* it is **d**ifferent of the transformed square).-

Fundamental Solutions of Magic Squares

The *fundamental solutions* of magic squares of order n, or of a special kind of that order; it is the *smallest set* of solutions of the which it is possible to obtain the total of solutions for that order or kind by means of *applicable symmetry operations on all solutions* of that order or kind.-

Pandiagonal Magic Squares of Order Four

Is well known that of the **880** magic squares of order four with the arithmetic progression of 1 to 16; **48** squares they are pandiagonal. The references about the number of *fundamental solutions* of these **48** pandiagonal squares, they deduce **3** squares.

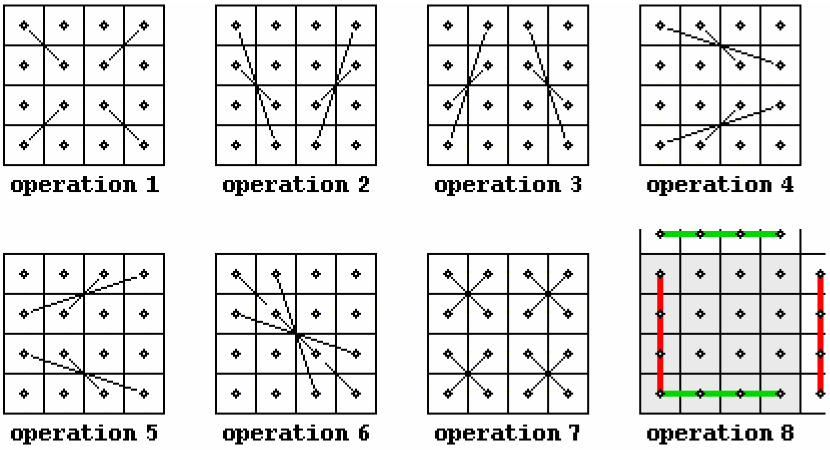
Why 3 ?

Because is well known the *symmetry operation* of a pandiagonal magic square in other pandiagonal magic squares by *rows and columns transposition*. With this type of operation, of **1** pandiagonal magic square of order four are obtained other **15** pandiagonal magic squares making a total of **16** squares, thus **3** = **48/16**. In other words, with **3** pandiagonal magic squares than it is necessary to determine them, are obtained the other 45 pandiagonal squares.

¿ But, is the rows and columns transposition the only applicable symmetry operation on all solutions for to obtain other pandiagonal magic squares of order four ?

The answer: Not

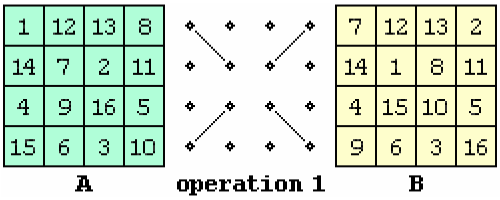
In the pandiagonal magic squares of order four I identify at least **8** *symmetry operations:*



In the context of this paper the `operation 8` is the *rows and columns transposition*, applied in the second step of the procedure. The `operation 7` is a common operation on all magic squares of order four and is equivalent at operations of transposition of rows and columns.-

Procedure

1) Of the `operations 1` to `6` can be chosen any of the following pairs of operations: `1-2; 1-3; 1-4; 1-5; 2-6; 3-6; 4-6` and `5-6;` and to apply them on any of the **48** pandiagonal magic squares; for example:

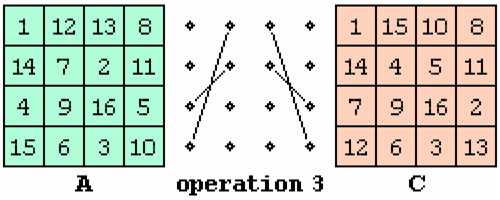


1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

A

7	12	13	2
14	1	8	11
4	15	10	5
9	6	3	16

B



1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

A

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

C

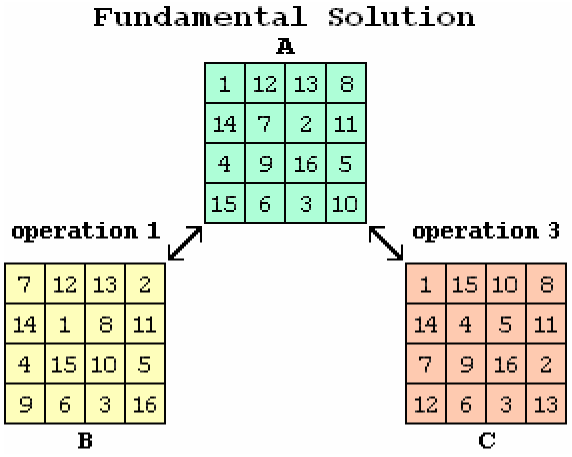
Remark: the pairs of operations `1-6; 2-3; 2-4; 2-5; 3-4; 3-5` and `4-5` are not consistent with the second step of the procedure.-

None of the above three squares can be obtained of any of the other two squares by the `operation 8` (*rows and columns transposition*), then:

2) Applying the `operation 8` on the squares **A**, **B** and **C** the **48** pandiagonal magic squares of order four are obtained.-

"As the squares **B** *and* **C** *are obtained by symmetry operations on the square* **A**; *this shows that the number of fundamental solutions is* **1** *square".-*

The 48 Pandiagonal Magic Squares of Order Four



Fundamental Solution

A

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

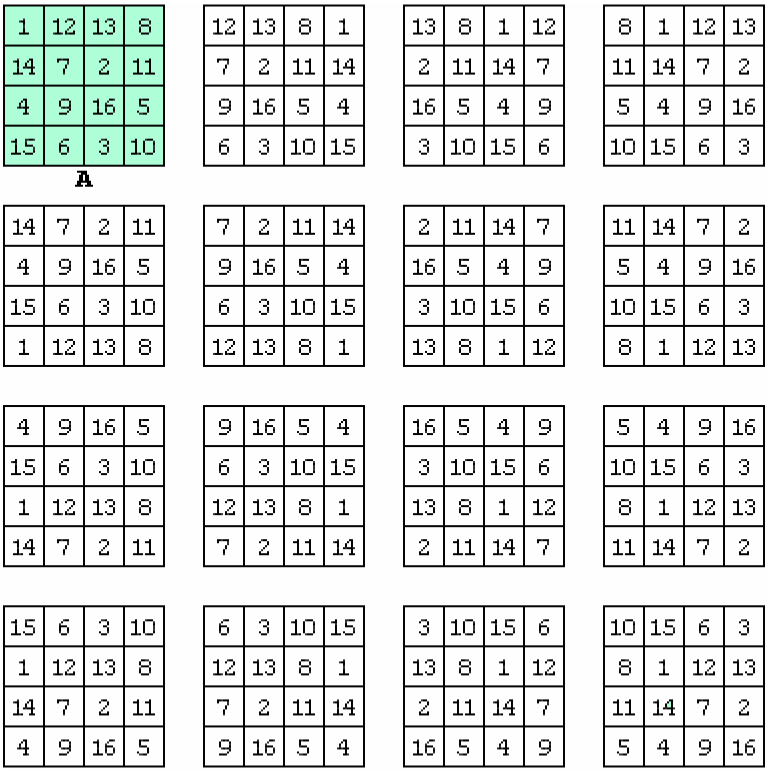
7	12	13	2
14	1	8	11
4	15	10	5
9	6	3	16

B

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

C

Operation 8 on A , B y C



1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

A

12	13	8	1
7	2	11	14
9	16	5	4
6	3	10	15

13	8	1	12
2	11	14	7
16	5	4	9
3	10	15	6

8	1	12	13
11	14	7	2
5	4	9	16
10	15	6	3

14	7	2	11
4	9	16	5
15	6	3	10
1	12	13	8

7	2	11	14
9	16	5	4
6	3	10	15
12	13	8	1

2	11	14	7
16	5	4	9
3	10	15	6
13	8	1	12

11	14	7	2
5	4	9	16
10	15	6	3
8	1	12	13

4	9	16	5
15	6	3	10
1	12	13	8
14	7	2	11

9	16	5	4
6	3	10	15
12	13	8	1
7	2	11	14

16	5	4	9
3	10	15	6
13	8	1	12
2	11	14	7

5	4	9	16
10	15	6	3
8	1	12	13
11	14	7	2

15	6	3	10
1	12	13	8
14	7	2	11
4	9	16	5

6	3	10	15
12	13	8	1
7	2	11	14
9	16	5	4

3	10	15	6
13	8	1	12
2	11	14	7
16	5	4	9

10	15	6	3
8	1	12	13
11	14	7	2
5	4	9	16

7	12	13	2
14	1	8	11
4	15	10	5
9	6	3	16

B

12	13	2	7
1	8	11	14
15	10	5	4
6	3	16	9

13	2	7	12
8	11	14	1
10	5	4	15
3	16	9	6

2	7	12	13
11	14	1	8
5	4	15	10
16	9	6	3

14	1	8	11
4	15	10	5
9	6	3	16
7	12	13	2

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

8	11	14	1
10	5	4	15
3	16	9	6
13	2	7	12

11	14	1	8
5	4	15	10
16	9	6	3
2	7	12	13

4	15	10	5
9	6	3	16
7	12	13	2
14	1	8	11

15	10	5	4
6	3	16	9
12	13	2	7
1	8	11	14

10	5	4	15
3	16	9	6
13	2	7	12
8	11	14	1

5	4	15	10
16	9	6	3
2	7	12	13
11	14	1	8

9	6	3	16
7	12	13	2
14	1	8	11
4	15	10	5

6	3	16	9
12	13	2	7
1	8	11	14
15	10	5	4

3	16	9	6
13	2	7	12
8	11	14	1
10	5	4	15

16	9	6	3
2	7	12	13
11	14	1	8
5	4	15	10

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

C

15	10	8	1
4	5	11	14
9	16	2	7
6	3	13	12

10	8	1	15
5	11	14	4
16	2	7	9
3	13	12	6

8	1	15	10
11	14	4	5
2	7	9	16
13	12	6	3

14	4	5	11
7	9	16	2
12	6	3	13
1	15	10	8

4	5	11	14
9	16	2	7
6	3	13	12
15	10	8	1

5	11	14	4
16	2	7	9
3	13	12	6
10	8	1	15

11	14	4	5
2	7	9	16
13	12	6	3
8	1	15	10

7	9	16	2
12	6	3	13
1	15	10	8
14	4	5	11

9	16	2	7
6	3	13	12
15	10	8	1
4	5	11	14

16	2	7	9
3	13	12	6
10	8	1	15
5	11	14	4

2	7	9	16
13	12	6	3
8	1	15	10
11	14	4	5

12	6	3	13
1	15	10	8
14	4	5	11
7	9	16	2

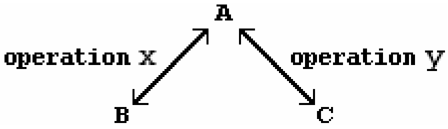
6	3	13	12
15	10	8	1
4	5	11	14
9	16	2	7

3	13	12	6
10	8	1	15
5	11	14	4
16	2	7	9

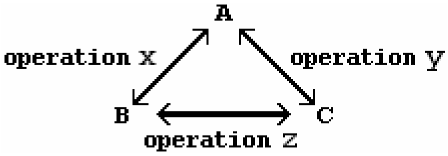
13	12	6	3
8	1	15	10
11	14	4	5
2	7	9	16

Closing the Triangle

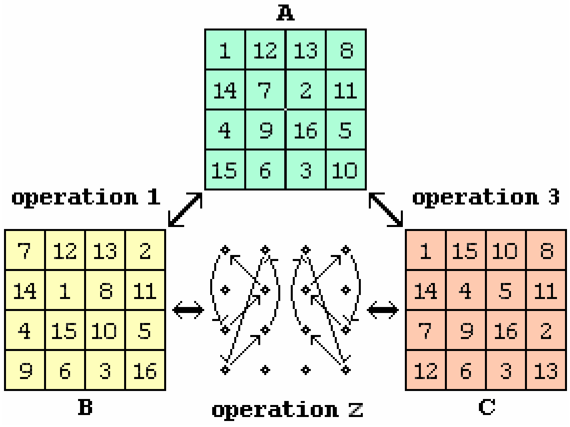
In the previous procedure we have:



¿ Is possible to close the triangle by means of a *symmetry operation* **z** *?* :



Sure, the existence of the **operation z** confers consistency to the procedure:



A

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

B

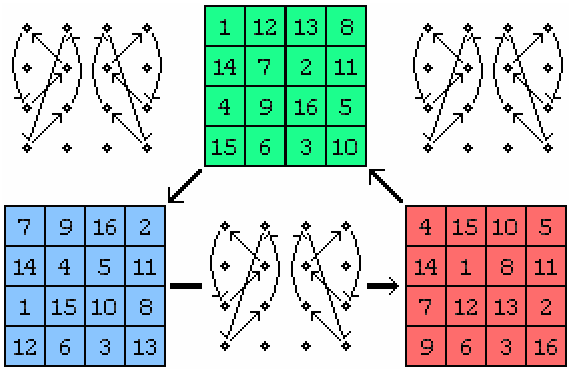
7	12	13	2
14	1	8	11
4	15	10	5
9	6	3	16

C

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

The **operation z** is commutative if the entries displacement is inverted (↻ | ↺).-
¿ This **operation z** between the squares **B** and **C**, is an exclusive operation among them *?* *Not, in any of their four orientations can be applied on any of the 48 pandiagonal magic squares.* If the entries displacement is not inverted, the triangle closure can be simplified *!!!* :

Three squares and one symmetry operation



1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

7	9	16	2
14	4	5	11
1	15	10	8
12	6	3	13

4	15	10	5
14	1	8	11
7	12	13	2
9	6	3	16

None of the above three squares can be obtained of any of the other two by rows and columns transposition, then the 48 squares also can be obtained with them.-

Although this type of symmetry operation by *entries displacement* is logical, it seem not to have been treated in the literature about magic squares.-

Pandiagonal Magic Squares of Order Five

Is well known that of the **275305224** magic squares of order five with the arithmetic progression of 1 to 25; **3600** squares they are pandiagonal. In general, the references about the number of *fundamental solutions* of these **3600** pandiagonal magic squares, they deduce **144** squares but Benson & Jacoby (1976) deduced **36** squares and **1** *"intermediate square"* (an algebraic structure). As 28800 variations of this algebraic structure are possible (this include rotations and reflections), all squares can be built with the "intermediate square" (28800/8 = 3600). This information is compiled with detail in the web page of Harvey Heinz … look it *!!!*

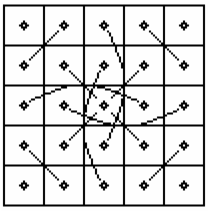
Why 144 *?*

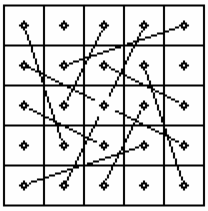
The same that in the pandiagonal magic squares of order four, because is well known the *symmetry operation* of a pandiagonal magic square in other pandiagonal magic squares by *rows and columns transposition*. With this type of operation, of **1** pandiagonal magic square of order five are obtained other **24** pandiagonal magic squares making a total of **25** squares, thus **144** = **3600**/**25**. In other words with **144** pandiagonal magic squares than is necessary to determine them, are obtained the other 3456 pandiagonal magic squares.

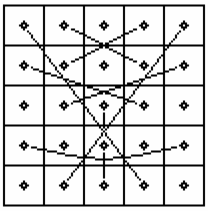
¿ But, is the rows and columns transposition the only applicable symmetry operation on all solutions for to obtain other pandiagonal magic squares of order five ?

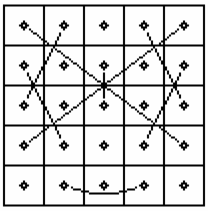
The answer: Not

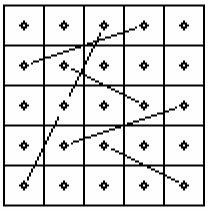
The total of *applicable symmetry operations on all* pandiagonal magic squares of order five it is unknown, but in the context of this paper is not necessary to know all. Some examples:

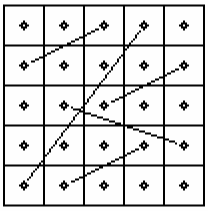

operation 1

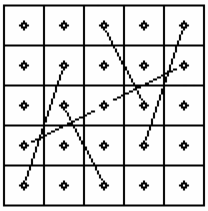

operation 2

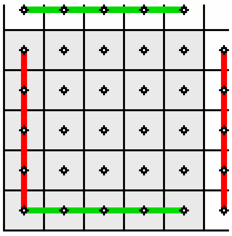

operation 3


operation 4


operation 5


operation 6


operation 7


operation 8

In the context of this paper the `operation 8` is the *rows and columns transposition*, applied in the third step of the procedure.

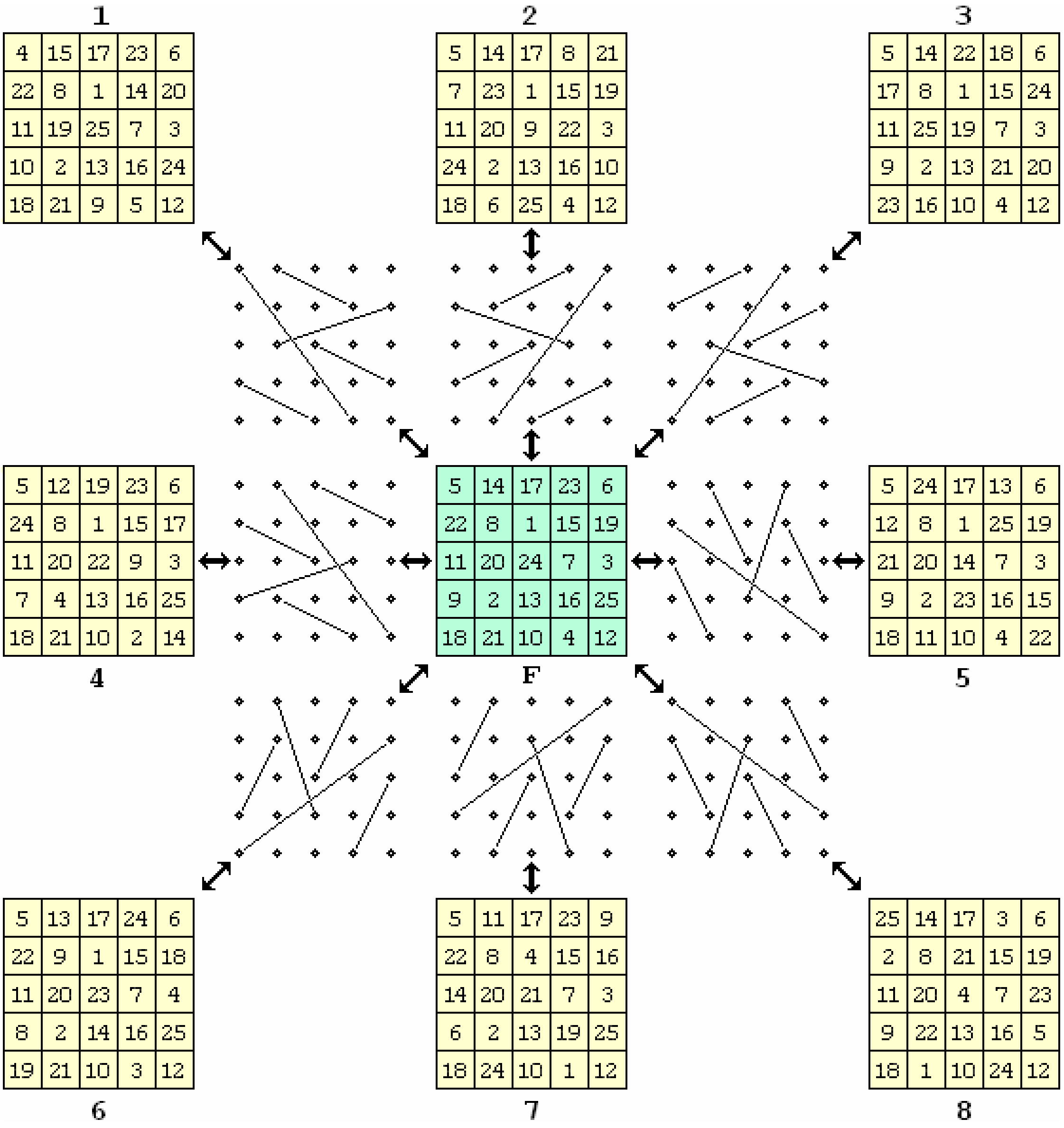
¿ Starting from **1** *pandiagonal magic square of order five is possible to obtain* **144** *pandiagonal magic squares of such way that none can be obtained of any of the others by rows and column transposition ?*

To obtain the **144** pandiagonal magic squares, in this paper is developed the following option:

1 x 9 x 16 = 144

Procedure

1) On any pandiagonal magic square of order five is applied the `operation 6` for to obtain other **8** pandiagonal magic squares as follows:



1

4	15	17	23	6
22	8	1	14	20
11	19	25	7	3
10	2	13	16	24
18	21	9	5	12

2

5	14	17	8	21
7	23	1	15	19
11	20	9	22	3
24	2	13	16	10
18	6	25	4	12

3

5	14	22	18	6
17	8	1	15	24
11	25	19	7	3
9	2	13	21	20
23	16	10	4	12

5	12	19	23	6
24	8	1	15	17
11	20	22	9	3
7	4	13	16	25
18	21	10	2	14

4

5	14	17	23	6
22	8	1	15	19
11	20	24	7	3
9	2	13	16	25
18	21	10	4	12

F

5	24	17	13	6
12	8	1	25	19
21	20	14	7	3
9	2	23	16	15
18	11	10	4	22

5

5	13	17	24	6
22	9	1	15	18
11	20	23	7	4
8	2	14	16	25
19	21	10	3	12

6

5	11	17	23	9
22	8	4	15	16
14	20	21	7	3
6	2	13	19	25
18	24	10	1	12

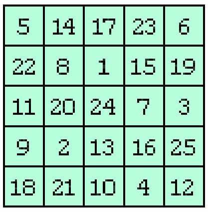
7

25	14	17	3	6
2	8	21	15	19
11	20	4	7	23
9	22	13	16	5
18	1	10	24	12

8

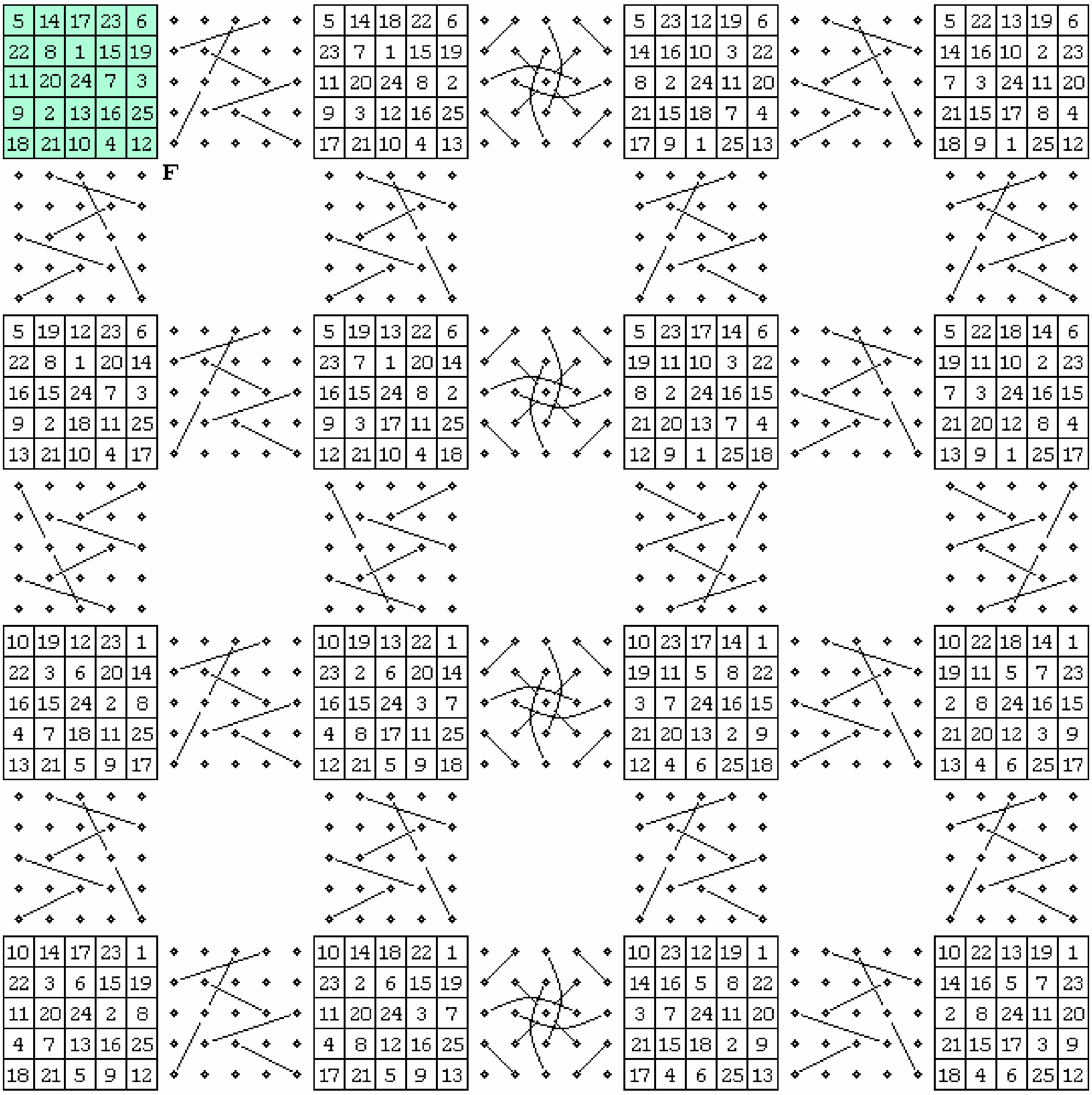
None of the **9** *squares can be obtained of any of the other* **8** *squares by rows and columns transposition.-*

2) Applying the `operations 1` and `5` as follows on each one of the above **9** squares, the **144** pandiagonal magic squares are obtained. As examples, will be shown the set of *symmetry operations* on the squares **F** and **6**:



5	14	17	23	6
22	8	1	15	19
11	20	24	7	3
9	2	13	16	25
18	21	10	4	12

F



5	14	17	23	6
22	8	1	15	19
11	20	24	7	3
9	2	13	16	25
18	21	10	4	12

F

5	14	18	22	6
23	7	1	15	19
11	20	24	8	2
9	3	12	16	25
17	21	10	4	13

5	23	12	19	6
14	16	10	3	22
8	2	24	11	20
21	15	18	7	4
17	9	1	25	13

5	22	13	19	6
14	16	10	2	23
7	3	24	11	20
21	15	17	8	4
18	9	1	25	12

5	19	12	23	6
22	8	1	20	14
16	15	24	7	3
9	2	18	11	25
13	21	10	4	17

5	19	13	22	6
23	7	1	20	14
16	15	24	8	2
9	3	17	11	25
12	21	10	4	18

5	23	17	14	6
19	11	10	3	22
8	2	24	16	15
21	20	13	7	4
12	9	1	25	18

5	22	18	14	6
19	11	10	2	23
7	3	24	16	15
21	20	12	8	4
13	9	1	25	17

10	19	12	23	1
22	3	6	20	14
16	15	24	2	8
4	7	18	11	25
13	21	5	9	17

10	19	13	22	1
23	2	6	20	14
16	15	24	3	7
4	8	17	11	25
12	21	5	9	18

10	23	17	14	1
19	11	5	8	22
3	7	24	16	15
21	20	13	2	9
12	4	6	25	18

10	22	18	14	1
19	11	5	7	23
2	8	24	16	15
21	20	12	3	9
13	4	6	25	17

10	14	17	23	1
22	3	6	15	19
11	20	24	2	8
4	7	13	16	25
18	21	5	9	12

10	14	18	22	1
23	2	6	15	19
11	20	24	3	7
4	8	12	16	25
17	21	5	9	13

10	23	12	19	1
14	16	5	8	22
3	7	24	11	20
21	15	18	2	9
17	4	6	25	13

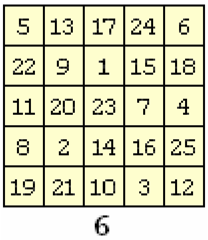
10	22	13	19	1
14	16	5	7	23
2	8	24	11	20
21	15	17	3	9
18	4	6	25	12

None of the **16** *squares can be obtained of any of the other* **15** *squares by rows and columns transposition.-*

To observe that:

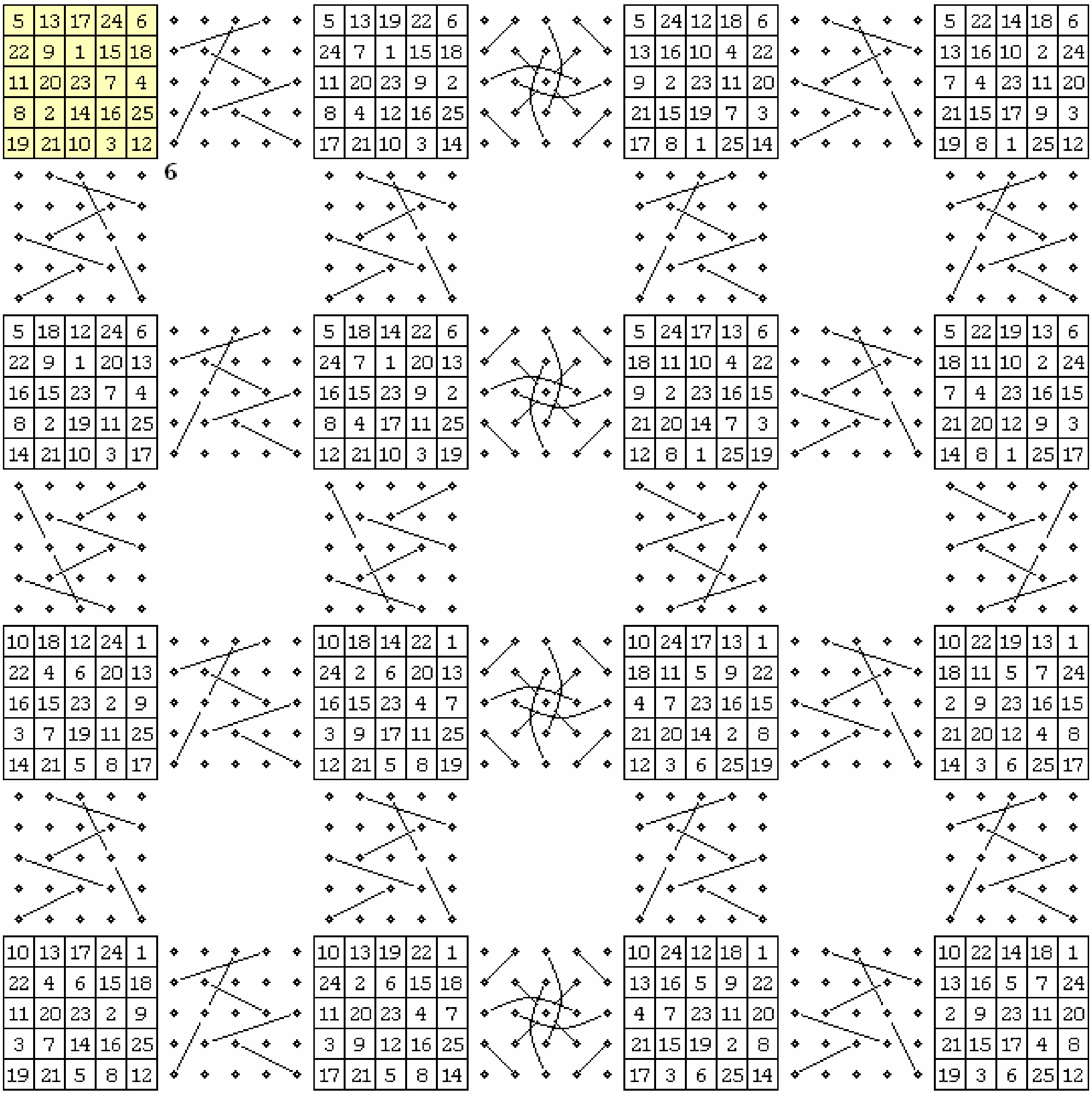
square (1, 1) ↔ `operation 1` ↔ square (1, 4)
square (1, 1) ↔ `operation 5` ↔ square (4, 1)

Then the total set can be considered as a *torus* of *matrices* and *symmetry operations.-*



5	13	17	24	6
22	9	1	15	18
11	20	23	7	4
8	2	14	16	25
19	21	10	3	12

6



5	13	17	24	6
22	9	1	15	18
11	20	23	7	4
8	2	14	16	25
19	21	10	3	12

5	13	19	22	6
24	7	1	15	18
11	20	23	9	2
8	4	12	16	25
17	21	10	3	14

5	24	12	18	6
13	16	10	4	22
9	2	23	11	20
21	15	19	7	3
17	8	1	25	14

5	22	14	18	6
13	16	10	2	24
7	4	23	11	20
21	15	17	9	3
19	8	1	25	12

5	18	12	24	6
22	9	1	20	13
16	15	23	7	4
8	2	19	11	25
14	21	10	3	17

5	18	14	22	6
24	7	1	20	13
16	15	23	9	2
8	4	17	11	25
12	21	10	3	19

5	24	17	13	6
18	11	10	4	22
9	2	23	16	15
21	20	14	7	3
12	8	1	25	19

5	22	19	13	6
18	11	10	2	24
7	4	23	16	15
21	20	12	9	3
14	8	1	25	17

10	18	12	24	1
22	4	6	20	13
16	15	23	2	9
3	7	19	11	25
14	21	5	8	17

10	18	14	22	1
24	2	6	20	13
16	15	23	4	7
3	9	17	11	25
12	21	5	8	19

10	24	17	13	1
18	11	5	9	22
4	7	23	16	15
21	20	14	2	8
12	3	6	25	19

10	22	19	13	1
18	11	5	7	24
2	9	23	16	15
21	20	12	4	8
14	3	6	25	17

10	13	17	24	1
22	4	6	15	18
11	20	23	2	9
3	7	14	16	25
19	21	5	8	12

10	13	19	22	1
24	2	6	15	18
11	20	23	4	7
3	9	12	16	25
17	21	5	8	14

10	24	12	18	1
13	16	5	9	22
4	7	23	11	20
21	15	19	2	8
17	3	6	25	14

10	22	14	18	1
13	16	5	7	24
2	9	23	11	20
21	15	17	4	8
19	3	6	25	12

None of the **16** *squares can be obtained of any of the other* **15** *squares, nor of the* ***15*** *derived squares of* **F** *by rows and columns transposition.-*

3) The same that in the pandiagonal magic squares of order four; applying the **`operation 8`** on the **144** squares**,** the **3600** pandiagonal magic squares of order five are obtained.

"As the **144** *squares are obtained by symmetry operations starting from the square* **`F`***; this shows that the number of fundamental solutions is* **1** *square".-*

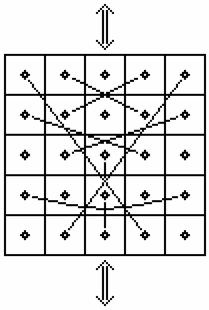
The "Intermediate Square"

A + a	B + b	C + c	D + d	E + e
C + d	D + e	E + a	A + b	B + c
E + b	A + c	B + d	C + e	D + a
B + e	C + a	D + b	E + c	A + d
D + c	E + d	A + e	B + a	C + b

This matrix it is the addition of two Latin Pandiagonal Squares and it is very useful for to verify if a *symmetry operation* it is applicable on all solutions.

For example, applying on the "intermediate square" the **`operation 3`**:

A + a	B + b	C + c	D + d	E + e
C + d	D + e	E + a	A + b	B + c
E + b	A + c	B + d	C + e	D + a
B + e	C + a	D + b	E + c	A + d
D + c	E + d	A + e	B + a	C + b



B + a	A + b	C + c	D + e	E + d
C + e	D + d	E + a	B + b	A + c
E + b	B + c	A + e	C + d	D + a
A + d	C + a	D + b	E + c	B + e
D + c	E + e	B + d	A + a	C + b

This matrix it is also the addition of two Latin Pandiagonal Squares, then the **`operation 3`** it is applicable on all solutions (the **`operation 3`** can be applied in their four orientations)

Conclusion

Whom deduces **3** *fundamental solutions* for the pandiagonal magic squares of order four and **144** *fundamental solutions* for the pandiagonal magic squares of order five as the *smallest set*, not deduces *symmetry operations* among them. If **3 x 16 = 48** and **144 x 25 = 3600** (**36 x 4 x 25 = 3600** Benson & Jacoby) they were explained; **1 x 3 x 16 = 48** and **1 x 9 x 16 x 25 = 3600** they were not explained**.** Now the procedure of to obtain all pandiagonal magic squares of order four and five starting from any pandiagonal magic square is simpler and this paper explains it.-

`Order Four: 1 x 3 x 16`
`Order Five: 1 x 9 x 16 x 25`

They seem two expressions of a sequence, but the pandiagonal magic squares of order n = 4k + 2 not exist. At the end of this paper a curiosity and nothing else: ¿ has some meaning **`Order Three 1 x 9`** …? Although *non-magic, they are the* **9** *pandiagonal squares of order three !!!* :

```
7 8 9     5 6 4     3 1 2
1 2 3     8 9 7     6 4 5
4 5 6     2 3 1     9 7 8

6 4 5     1 2 3     8 9 7
9 7 8     4 5 6     2 3 1
3 1 2     7 8 9     5 6 4

2 3 1     9 7 8     4 5 6
5 6 4     3 1 2     7 8 9
8 9 7     6 4 5     1 2 3
```

References:

- Suzuki Mutsumi:
  "Complete Magic Squares of 4x4 (48 =3 x 16)", **199**_?.-
  "Complete Magic Squares (Pan Magic Squares) of 5x5 (144x25)", **199**_?.-
- Grogono Alan: http://www.grogono.com/magic/
  "The 4x4 Pan-Magic Squares", **1998**.-
  "The 5x5 Pan-Magic Squares", **199**_?.-
- Karl Christoph: *"The Magic Square 4x4"*, **1999**.-
- Heinz Harvey: http://www.magic-squares.net/pandiag5.htm#It%20gets%20better
  "Pandiagonal 5x5". **2001**.-
- Setsuda Kanji: http://homepage2.nifty.com/KanjiSetsuda/pages/EnglishP1.html
  "Transformations and Fundamental Solutions of Pan-Magic Squares 4x4", 2001-**2003**.-
  "Algebraic Study of Composite Magic Squares 4x4", 2001-**2005**.-
  "New Study of Pandiagonal Magic Squares of Order 5", 2001-**2005**.-
  "New Algebraic Study of Pan-Magic Squares 4x4", 2002-**2005**.-
- Amela Miguel:
  "Total de Cuadrados Mágicos 4x4 [880]". **1991**.-
  "Total de Cuadrados Mágicos 5x5 [275305224]: La Constante Concéntrica". **2003**.-
  "Modèles Structurels de Carrés Magiques 4x4",**2009**.-
  "Total Compactness in Magic Squares of Order Five",**2010**.-