ON MAGIC AND CONSECUTIVE LABELINGS OF PLANE GRAPHS

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ABSTRACT. We define the notions of magic and consecutive labelings of plane graphs. Magic labelings are often constructed from complementary consecutive labelings. We extend some magic and complementary consecutive labelings of an almost forgotten Chinese amateur mathematician Pao Chhi-Shou (c. 1880) to families of plane graphs, including wheels, friendship graphs, and prisms.

1. *Introduction.*

Graph labelings have lately aroused considerable attention. They gave birth to families of graphs with attractive names such as magic, elegant, graceful, and harmonious. They exhibited the delicacy of combinatorial constructions and promised interesting applications. Browsing through the literature, for example [1, 2, 3, 4, 5, 6, 7, 8, 11, 12], one becomes at least a nodding acquaintance with these beauties. Usually labelings are done on vertices of a graph so that conditions on edges are met. However, when a graph is plane, i.e., it has been laid out on the Euclidean plane with crossings of edges only at vertices, it is natural to impose requirements on faces. Moreover, a liberal policy even condones the extension of labels to edges and faces. The simplest choice for a requirement would be equal weight (sum of vertex, edge, and face labels) for faces with equal number of sides. It came as a surprise that such constructions already appeared in a treatise published in A.D. 1275 by the Chinese mathematician Yang Hui. Here and subsequently, we depend on the authoritative work of Li Nien [9] for historical data. Surely, Yang

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did not have the concept of a graph. Nonetheless, in an article on magic squares, Yang extended magic constructions to plane configurations some of which fall naturally under the concept just mentioned. This theme was further developed by Chang Chhao circa A.D. 1670. It finally reached the amazing achievements of an almost unknown amateur mathematician Pao Chhi-Shou, circa A.D. 1880, of constructing magic labelings for the platonic polyhedra and icosidodecahedron. (See section 6.) Pao followed the traditional custom of not revealing the methods by which he obtained his results. Unfortunately, except labelings for the cube, his constructions were illustrated largely by plane net representations of polyhedra. In view of the appearance of repeated labels, their true significance was commonly overlooked. For example, Needham [10, fig. 62, p. 60] shows only one of his easier products. In this paper, we are going to clarify the concepts behind Pao's labelings and extend them to families of plane graphs, including wheels, friendship graphs, and prisms.

2. *Basic notions and examples.*

A graph G consists of a vertex set V and an edge set E with cardinalities v and e, respectively. We only consider graphs without loops and multiple edges. A graph is said to be *plane* if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph. We make the convention that all plane graphs considered in this paper possess no end vertices, i.e., vertices of degree one. For a plane graph G, it makes sense to determine its faces, including the unique face of infinite area. Let f be the number of faces of G. A *labeling of type (a, b, c)* assigns labels from the set $\{1, 2, 3, \ldots, av + be + cf\}$ to the vertices, edges, and faces of G such that each vertex receives a labels, each edge receives b labels, and each face receives c labels and each number is used exactly once as a label. On most occasions, we restrict a and c to be no greater than one. Labelings of types (1, 0, 0) and (0, 1, 0) are also called vertex and edge labelings, respectively. The *weight* of a face under a labeling is the sum of labels of the face itself together with labels of vertices and edges surrounding that face. A labeling is said to be *magic*, if for every number s, all s-side faces have the same weight. We allow different weights for different s. This notion of being magic is entirely different from those defined in [4, 7, 12]. A labeling is said to be *consecutive* if, for every number s, the weights of all s-side faces constitute a set of consecutive integers. We allow different sets for different s. However, we do not consider the mixed type of some being equal and some being consecutive. Two labelings L and L' are said to be *complementary* if, for every number s, the sum of L-weight and L'-weight of each s-side face is a constant depending on s.

There are two obvious methods of producing new magic labelings from old ones.

(1) If G has two magic or complementary consecutive labelings of types (a, b, c) and (a', b', c'), then G has a magic labeling of type (a+a', b+b', c+c').

Add av + be + cf to each label of the second labeling. Then combine two labelings into one.

(2) If G has a magic labeling of type (a, b, c), then G has magic labelings of types (a+2k, b+2k', c+2k") for all choices of natural numbers k, k', and k".

The numbers 1, 2, 3, ..., 2v can be paired off into (1, 2v), (2, 2v-1), ..., (v, v+1). Add av + be + cf to each number. Then place each pair as further labels of a vertex. The vertex type is so increased by 2. This procedure can be repeated and applied to edges and faces to increase types by multiples of 2.

Now we illustrate various notions by applying them to the graph of the simplest platonic regular polyhedron, tetrahedron. Its graph is the complete graph on four vertices. Referring to figure 1, (a) and (b) give magic labelings of types (1, 1, 0) and (0, 1, 1). The consecutive labeling in (c) together with its obvious complementary face labeling will give a magic labeling of type (1, 0, 1). The consecutive labeling of (d) will entail a magic labeling of type (1, 1, 1). Magic labelings of types (1, 2k, 0) can be generated from (e) and of types (1, 2k+1, 0) from (a). It is easy to see that there are no magic labelings of types (1, 0, 0), (0, 1, 0), and (0, 0, 1). Consecutive edge labelings do not exist either. We finally remark that (c) and (e) were Pao's discoveries.

3. *Wheels.*

For $n \geq 3$, the *wheel* W_n is a graph consisting of an n-side polygon and a center vertex which is adjacent to every other vertex. Let $a_1, a_2, \ldots, a_n$ denote the n vertices of the polygon in the counter-clockwise direction and a_{n+1} denote the center vertex. W_3 is the graph of tetrahedron. Obviously, wheels do not have magic vertex labelings.

THEOREM 1. *For $n \geq 3$, the wheel W_n has a consecutive vertex labeling if and only if $n \not\equiv 2 \pmod 4$.*

Proof. Case 1. $n = 2k+1$.

Construction of L_1 :

$$L_1(a_i) = \begin{cases} k + (i+1)/2 & \text{if } i \text{ is odd,} \\ i/2 & \text{if } i \text{ is even,} \\ n+1 & \text{if } i = n+1. \end{cases}$$

Verification: The weights successively take the values $3k+4, 3k+5, \ldots, 5k+4$. When $n = 3$, we have to check that the weight of the infinite face, $L_1(a_1) + L_1(a_2) + L_1(a_3) = 6$, does not violate the consecutive property.

Case 2. $n = 4K$.

Construction of L_2 :

$$L_2(a_i) = \begin{cases} (i+1)/2 & \text{if } i \leq 2k-1 \text{ is odd,} \\ (i+3)/2 & \text{if } 2k+1 \leq i \leq n-1 \text{ is odd,} \\ 2k+1 + (i/2) & \text{if } i \text{ is even,} \\ k+1 & \text{if } i = n+1. \end{cases}$$

Verification: $L_2(a_i) + L_2(a_{i+1}) + L_2(a_{n+1}) = 3k + 3 + i$ for $i = 1, 2, \ldots, 2k-1$; $L_2(a_1) + L_2(a_n) + L_2(a_{n+1}) = 5k + 3$; $L_2(a_{2k}) + L_2(a_{2k+1}) + L_2(a_{n+1}) = 5k + 4$; $L_2(a_i) + L_2(a_{i+1}) + L_2(a_{n+1}) = 3k + 4 + i$ for $i = 2k+1, 2k+2, \ldots, n-1$.

Case 3. $n = 4k + 2$.

Suppose the number j is placed at the center and consecutive weights begin with x. We have $nx + ((n-1)n/2) = nj + 2((n+1)(n+2)/2 - j)$, hence $n^2 + 7n + 4 = 2(nx - (n-2)j)$. Substituting $4k + 2$ for n, we get $8k^2 + 22k + 11 = 2((2k+1)x - 2kj)$ which is impossible in parity. So there is no consecutive vertex labeling in this case.

Figure 2 shows consecutive vertex labelings L_1 and L_2 for W_7 and W_8, respectively.

THEOREM 2. *For $n > 3$, the wheel W_n has a consecutive edge labeling. Furthermore, when $n \not\equiv 2 \pmod 4$, this labeling can be made to be complementary to the consecutive vertex labeling obtained in Theorem 1.*

Proof. Define the edge labeling L_3 as follows.

$$L_3(a_ia_j) = \begin{cases} n+1-i & \text{if } j = n+1, \\ n+1+i & \text{if } i \le n-1 \text{ and } j = i+1, \\ n+1 & \text{if } i = n \text{ and } j = 1. \end{cases}$$

It is easy to see that the weights successively take the values 3n+1, 3n, ..., 2n+2. Hence L_3 is complementary to L_1 when n is odd. For the case $n = 4k$, we have to construct another consecutive edge labeling L_4 complementary to L_2.

Construction of L_4 :

$$L_4(a_ia_j) = \begin{cases} 6k+1+i & \text{if } i \le 2k-1 \text{ and } j = i+1, \\ 6k+1 & \text{if } i = 2k \text{ and } j = i+1, \\ 2k+i & \text{if either } i = n \text{ and } j = 1 \\ & \text{or } 2k+1 \le i \le n-1 \text{ and } j = i+1, \\ 4k+1-i & \text{if } i \le 2k-1 \text{ is odd and } j = n+1, \\ 6k-i & \text{if } i \ge 2k+1 \text{ is odd and } j = n+1, \\ 2k+1-i & \text{if } i \le 2k \text{ is even and } j = n+1, \\ 4k+2-i & \text{if } i \ge 2k+2 \text{ is even and } j = n+1. \end{cases}$$

Verification: $L_4(a_ia_{i+1}) + L_4(a_ia_{n+1}) + L_4(a_{i+1}a_{n+1}) = 12k + 2 - i$ for $i = 1, 2, \ldots, 2k-1$; $L_4(a_na_1) + L_4(a_na_{n+1}) + L_4(a_1a_{n+1}) = 10k + 2$; $L_4(a_ia_{i+1}) + L_4(a_ia_{n+1}) + L_4(a_{i+1}a_{n+1}) = 12k + 1 - i$ for $i = 2k, 2k+1, \ldots, n-1$.

Figure 3 shows consecutive edge labelings L_3 and L_4 for W_7 and W_8, respectively.

4. Friendship graphs.

The *friendship graph* F_n is a set of n triangles having a common center vertex. Let c denote the center vertex. For the ith triangle, let a_i and b_i denote the other two vertices.

THEOREM 3. *For $n > 1$, the friendship graph F_n has a consecutive vertex labeling.*

Proof. Case 1. $n = 2k+1$.

Construction of L_5 :

$$L_5(a_i) = i+1 \quad \text{for all } i.$$

$$L_5(b_i) = \begin{cases} 3k+2+i & \text{if } i \le k+1, \\ k+1+i & \text{if } k+2 \le i \le n. \end{cases}$$

$$L_5(c) = 1.$$

Verification: $L_5(a_i) + L_5(b_i) + L_5(c)$ is equal to $3k + 4 + 2i$ for $i = 1, 2, \ldots, k+1$ and is equal to $k + 3 + 2i$ for $i = k+2, k+3, \ldots, n$, i.e., $3k + 5 + 2j$ for $j = 1, 2, \ldots, k$.

Case 2. $n = 2k$.

Construction of L_6 :

$$L_6(a_i) = \begin{cases} i & \text{if } i \le k, \\ i+1 & \text{if } k+1 \le i \le n. \end{cases}$$

$$L_6(b_i) = \begin{cases} 3k+1+i & \text{if } i \le k, \\ k+1+i & \text{if } k+1 \le i \le n. \end{cases}$$

$$L_6(c) = k+1.$$

Verification: $L_6(a_i) + L_6(b_i) + L_6(c)$ is equal to $4k + 2 + 2i$ for $i = 1, 2, \ldots, k$ and is equal to $2k + 3 + 2i$ for $i = k+1, k+2, \ldots, n$, i.e., $4k + 3 + 2j$ for $j = 1, 2, \ldots, k$.

Figure 4 shows consecutive vertex labelings L_5 and L_6 for F_5 and F_6, respectively.

THEOREM 4. *For $n > 1$, the friendship graph F_n has a consecutive edge labeling complementary to the consecutive vertex labeling obtained in Theorem 3.*

Proof. Since each label will contribute to the weight of exactly one face, this labeling is rather easy to find. Split the numbers 1, 2, ..., 3n into triples (1,2n,3n), (2,2n-1,3n-1), (3,2n-2,3n-2), ..., (n,n+1,2n+1). They produce consecutive weights 5n+1, 5n, 5n-1, ..., 4n+2. Arranging these triples on the edges of triangles appropriately, we get a consecutive edge labeling complementary to L_5 or L_6.

5. *Prisms.*

For $n \geq 3$, the *prism* R_n is the cartesian product $P_2 \times C_n$ of a path of length 2 and a cycle of length n. Let $a_1, a_2, \ldots, a_n$ denote vertices on the outer cycle in the counterclockwise direction and $b_1, b_2, \ldots, b_n$ denote vertices on the inner cycle such that a_i and b_i are adjacent. We also make the convention that $a_{n+1} = a_1$ and $b_{n+1} = b_1$ to simplify later notation. R_4 is the graph of the cube.

THEOREM 5. *If $n \geq 4$ is even, then the prism R_n has a magic vertex labeling.*

Proof. Construction of L_7:

$$L_7(a_i) = \begin{cases} i & \text{if } i \text{ is odd,} \\ 2n-i & \text{if } i < n \text{ is even,} \\ 2n & \text{if } i = n. \end{cases}$$

$$L_7(b_i) = \begin{cases} 2n-i & \text{if } i \text{ is odd,} \\ i+2 & \text{if } i < n \text{ is even,} \\ 2 & \text{if } i = n. \end{cases}$$

Verification: It is easy to see that the common weight for all 4-side faces is $8k+2$. By subtracting similar terms of the weights of the two n-side faces, we can see that their difference is zero.

Figure 5 shows the magic vertex labeling L_7 of R_8.

THEOREM 6. *If* $n = 2k + 1 \geq 3$, *then the prism* R_n *has a consecutive vertex labeling.*

Proof. Case 1. k is odd.

Construction of L_8:

$$L_8(a_i) = \begin{cases} (i+1)/2 & \text{if } i \leq k \text{ is odd,} \\ 2n+1-i & \text{if } i \geq k+2 \text{ is odd or} \\ & \phantom{\text{if }} 1 \leq k+1 \text{ is even,} \\ k+1+(i/2) & \text{if } i \geq k+3 \text{ is even.} \end{cases}$$

$$L_8(b_i) = \begin{cases} 2n+1-i & \text{if } i \leq k \text{ is odd or} \\ & \phantom{\text{if }} i \geq k+3 \text{ is even,} \\ (i+1)/2 & \text{if } i \geq k+2 \text{ is odd,} \\ k+1+(i/2) & \text{if } i \leq k+1 \text{ is even.} \end{cases}$$

Verification: $L_8(a_i) + L_8(b_i) + L_8(a_{i+1}) + L_8(b_{i+1}) = 9k + 7 - i$ for all i. The difference between the sum of all b-labels and the sum of all a-labels is 1.

Case 2. k is even.

Construction of L_9:

$$L_9(a_i) = \begin{cases} 2n+1-i & \text{if } i \leq n-2 \text{ is odd,} \\ k+1 & \text{if } i = n, \\ i/2 & \text{if } i \leq k \text{ is even,} \\ k+1+(i/2) & \text{if } i \geq k+2 \text{ is even.} \end{cases}$$

$$L_9(b_i) = \begin{cases} k+1+((i+1)/2) & \text{if } i \leq k-1 \text{ is odd,} \\ (i+1)/2 & \text{if } k+1 \leq i \leq n-2 \text{ is odd,} \\ n+1 & \text{if } i = n, \\ 2n+1-i & \text{if } i \text{ is even.} \end{cases}$$

Verification: $L_9(a_i) + L_9(b_i) + L_9(a_{i+1}) + L_9(b_{i+1}) = 9k + 7 - i$ except $i = k$ and $i = n$. $L_9(a_1) + L_9(b_1) + L_9(a_n) + L_9(b_n) = 8k + 7$ and $L_9(a_k) + L_9(b_k) + L_9(a_{k+1}) + L_9(b_{k+1}) = 7k + 6$. The difference between the sum of all b-labels and the sum of all a-labels is 1.

Figure 6 shows consecutive vertex labelings L_8 and L_9 of R_7 and R_9, respectively.

THEOREM 7. *If $n = 2k \geq 4$, then the prism R_n has a magic edge labeling.*

Proof. Construction of L_{10}: Labels of R_4 are defined as shown in figure 7(a). Now assume $k \geq 3$.

$$L_{10}(a_ia_{i+1}) = \begin{cases} 3n+1-3i & \text{if } i \leq k-2, \\ 3k+3 & \text{if } i = k-1, \\ 3k-3 & \text{if } i = k, \\ 3i-3k & \text{if } k+1 \leq i \leq n-2, \\ 3k-2 & \text{if } i = n-1, \\ 3k+4 & \text{if } i = n. \end{cases}$$

$$L_{10}(b_ib_{i+1}) = \begin{cases} 2 & \text{if } i = 1, \\ 3n-3i & \text{if } 2 \leq i \leq k-2, \\ 3k+2 & \text{if } i = k-1, \\ 6k & \text{if } i = k, \\ 3n-1 & \text{if } i = k+1, \\ 3i+1-3k & \text{if } k+2 \leq i \leq n-2, \\ 3k-1 & \text{if } i = n-1, \\ 1 & \text{if } i = n. \end{cases}$$

$$L_{10}(a_ib_i) = \begin{cases} 3n-3 & \text{if } i = 1, \\ 3i-1 & \text{if } 2 \leq i \leq k-1, \\ 3k+1 & \text{if } i = k, \\ 4 & \text{if } i = k+1, \\ 9k+2-3i & \text{if } k+2 \leq i \leq n-1, \\ 3k & \text{if } i = n. \end{cases}$$

Verification: The common weight for all 4-side faces is $6n+2$. Each pair of numbers j and $3n-j+1$ is placed simultaneously either on a-cycle or on b-cycle. So the two n-side faces have the same weight.

Figure 7(b) shows the magic labeling L_{10} of R_8. We remark that L_7 and L_{10} labelings of R_4 were discovered by Pao.

THEOREM 8. *If* $n = 2k+1 \geq 3$, *then the prism* R_n *has a consecutive edge labeling complementary to the consecutive vertex labeling obtained in Theorem 6.*

Proof. Case 1. k is odd.

Construction of L_{11}:

$$L_{11}(a_i a_{i+1}) = \begin{cases} 5k+3+i & \text{if } i \leq k \text{ is odd,} \\ 3k+2+i & \text{if } k+2 \leq i \leq n-2 \text{ is odd,} \\ 2n & \text{if } i = n, \\ 2n-2i & \text{if } i \leq k-1 \text{ is even,} \\ 3n-2i & \text{if } k+1 \leq i \leq n-1 \text{ is even.} \end{cases}$$

$$L_{11}(b_i b_{i+1}) = \begin{cases} 2n-2i & \text{if } i \leq k \text{ is odd,} \\ 3n-2i & \text{if } k+2 \leq i \leq n-2 \text{ is odd,} \\ 5k+3 & \text{if } i = n, \\ 5k+3+i & \text{if } i \leq k-1 \text{ is even,} \\ 3k+2+i & \text{if } k+1 \leq i \leq n-1 \text{ is even.} \end{cases}$$

$$L_{11}(a_i b_i) = i \quad \text{for all } i.$$

Verification: The weights of the 4-side faces successively take the values $9k+7$, $9k+8$, ..., $11k+7$. The difference between the sum of all a-labels and the sum of all b-labels is 1.

Figure 8 shows that consecutive edge labeling L_{11} of R_7.

Case 2. k is even.

In order to obtain a consecutive edge labeling complementary to L_9, we first construct an auxiliary labeling L_{12}. All weights of 4-side faces will be consecutive under L_{12}.

Construction of L_{12}:

$$L_{12}(a_i a_{i+1}) = \begin{cases} 5k+3+i & \text{if } i \leq k-1 \text{ is odd,} \\ 2n & \text{if } i = k+1, \\ 3n-2i & \text{if } k+3 \leq i \leq n-2 \text{ is odd,} \\ n+1 & \text{if } i = n, \\ 2n-2i & \text{if } i \leq k-2 \text{ is even,} \\ 2n+1 & \text{if } i = k, \\ 3k+2+i & \text{if } k+2 \leq i \leq n-1 \text{ is even.} \end{cases}$$

$$L_{12}(b_ib_{i+1}) = \begin{cases} 2n-2i & \text{if } i \le k-1 \text{ is odd,} \\ 2n-1 & \text{if } i = k+1, \\ 3k+2+i & \text{if } k+3 \le i \le n-2 \text{ is odd,} \\ 3n & \text{if } i = n, \\ 5k+3+i & \text{if } i \le k-2 \text{ is even,} \\ 5k+3 & \text{if } i = k, \\ 3n-2i & \text{if } k+2 \le i \le n-1 \text{ is even.} \end{cases}$$

$$L_{12}(a_ib_i) = \begin{cases} i & \text{if } i \le k+1 \\ i+1 & \text{if } k+2 \le i \le n-1 \text{ is even,} \\ i-1 & \text{if } k+3 \le i \le n \text{ is odd.} \end{cases}$$

Verification: $L_{12}(a_ia_{i+1}) + L_{12}(b_ib_{i+1}) + L_{12}(a_ib_i) + L_{12}(a_{i+1}b_{i+1}) = 9k + 6 + i$ except $i = k$ and $i = n$. $L_{12}(a_na_1) + L_{12}(b_nb_1) + L_{12}(a_nb_n) + L_{12}(a_1b_1) = 10k + 6$ and $L_{12}(a_ka_{k+1}) + L_{12}(b_kb_{k+1}) + L_{12}(a_kb_k) + L_{12}(a_{k+1}b_{k+1}) = 11k + 7$. So the weights of 4-side faces are consecutive and complementary to L_9. However the sum of all b-labels is greater than the sum of all a-labels by $k-3$. We define L_{13} by switching some labels of a_ia_{i+1} with labels of b_ib_{i+1} under the L_{12} labeling to make the sum of all a-labels greater than the sum of all b-labels by 1.

Switching Rules for L_{13}: For $k = 2$ and $k = 4$, we have to construct them directly as shown in Figure 9. Now we assume $k = 2m \ge 6$.

Case 1. $m \equiv 1 \pmod 3$.

Let $j = (m-1)/3$. Switch labels of a_ia_{i+1} with labels of b_ib_{i+1} for $i = 1, 2, \ldots, 2j$.

Case 2. $m \equiv 0 \pmod 3$.

Let $j = m/3$. Switch labels of a_ia_{i+1} with labels of b_ib_{i+1} for $i = 1, 2, \ldots, 2j$ and $i = k+1$.

Case 3. $m \equiv 2 \pmod 3$.

Let $j = (m+4)/3$. Switch labels of a_ia_{i+1} with labels of b_ib_{i+1} for $i = 1, 2, \ldots, 2j$ and $i = k+2$.

By switching labels of a_ia_{i+1} and $a_{i+1}a_{i+2}$ with labels of b_ib_{i+1} and $b_{i+1}b_{i+2}$, we take out 3 points from the inner cycle. The adjustment at $i = k+1$ or $i = k+2$ will give back 1 or 5 points. Thus the desired result follows. Figure 10 shows R_{13} under labelings

L_{12} and L_{13}, respectively.

6. *The rest of the platonic family and a loner.*

In this section, we are going to show some nice labelings of the graphs of octahedron, dodecahedron, icosahedron, and icosidodecahedron. Unless noted otherwise, these are due to Pao Chhi-Shou. Because the Chinese sources are inaccessible to most readers, we have chosen Pao's more difficult labelings and drawn them in modern forms.

THEOREM 9. *The graph of octahedron has complementary consecutive vertex and edge labelings.*

Proof. See Figure 11. The edge labeling was discovered by us.

THEOREM 10. *The graph of dodecahedron has complementary consecutive vertex and edge labelings.*

Proof. See Figure 12.

THEOREM 11. *The graph of icosahedron has a consecutive vertex labeling.*

Proof. See Figure 13.

This vertex labeling was discovered by us. However, we have failed to determine whether it has a complementary consecutive edge labeling.

We only draw vertex labels in the final three figures to avoid crowding of labels. We list edge labels in the text by the following devise. Let $(i,j;\ k_1,k_2,\ldots,k_n)$ denote that the labels $k_1,\ k_2,\ \ldots,\ k_n$ are assigned to the edge with end vertices labeled by i and j.

Figure 14 shows the vertex labels of a type (1, 2, 0) magic labeling of icosahedron. The edge labels are as follows.

(1, 8;25,54)
(1, 9;34,59)
(1,11;24,57)
(1,13;39,55)
(1,15;33,56)

(3, 7;23,64)

(3, 8;19,65)

(3,12; 6,67)

(3,13;42,45)

(3,14;31,62)

(5, 7;27,63)

(5, 9;32,60)

(5,10;43,46)

(5,14;18,70)

(5,15;20,66)

(7, 8;37,53)

(7, 9;36,40)

(7,14; 4,71)

(8, 9;41,48)

(8,13;16,68)

(9,15;22,50)

(10,11;29,58)

(10,12;17,69)

(10,14;26,47)

(10,15; 2,72)

(11,12;21,52)

(11,13;28,51)

(11,15;38,44)

(12,13;30,61)

(12,14;35,49)

Figure 15 shows the vertex labels of a type (1, 3, 0) magic labeling of icosahedron. The edge labels are as follows.

(1, 5;29,58, 91)

(1, 6;26,54, 93)

(1, 8;42,47, 82)

(1,10;37,46, 89)

(1,12;18,59,101)

(2, 4;15,62,100)

(2, 5;27,67, 81)

(2, 9;40,51, 86)

(2,10;24,70, 80)
(2,11;20,69, 79)
(3, 4;23,72, 75)
(3, 6;13,65,102)
(3, 7;32,68, 74)
(3,11;21,61, 92)
(3,12;28,57, 84)
(4, 5;25,55, 94)
(4, 6;14,64, 96)
(4,11;17,60, 98)
(5, 6;16,71, 87)
(5,10;33,50, 88)
(6,12;35,49, 83)
(7, 8;38,48, 85)
(7, 9;19,63, 90)
(7,11;30,39, 99)
(7,12;43,53, 76)
(8, 9;31,66, 73)
(8,10;22,56, 97)
(8,12;44,45, 78)
(9,10;34,36, 95)
(9,11;41,52, 77)

Figure 16 shows the vertex labels of a type (1, 1, 0) magic labeling of icosidodecahedron. The edge labels are as follows.

(1, 3;75)	(11,15;53)
(1, 4;72)	(11,16;52)
(1,26;35)	(11,23;78)
(1,29;90)	(11,28;38)
(2, 4;74)	(12,14;55)
(2, 5;71)	(12,15;54)
(2,27;34)	(12,25;76)
(2,30;89)	(12,30;36)
(3, 5;73)	(13,14;51)
(3,26;88)	(13,20;45)
(3,28;33)	(13,22;79)

(4,27;87)	(13,27;39)
(4,29;32)	(14,22;49)
(5,28;86)	(14,30;81)
(5,30;31)	(15,25;46)
(6, 8;68)	(15,28;83)
(6, 9;70)	(16,17;44)
(6,21;65)	(16,23;48)
(6,23;60)	(16,26;85)
(7, 9;67)	(17,18;42)
(7,10;69)	(17,21;80)
(7,22;64)	(17,26;40)
(7,24;59)	(18,19;43)
(8,10;66)	(18,21;50)
(8,23;63)	(18,29;82)
(8,25;58)	(19,20;41)
(9,21;57)	(19,24;77)
(9,24;62)	(19,29;37)
(10,22;56)	(20,24;47)
(10,25;61)	(20,27;84)

7. *Conclusion.*

We have introduced the notions of magic and consecutive labelings of type (a, b, c) for plane graphs. Complementary consecutive labelings played interesting roles in obtaining magic labelings. We have extended Pao Chhi-Shou's classical labelings of platonic polyhedra to families of plane graphs including wheels, friendship graphs, and prisms. It seems that quite a number of plane graphs would possess similar labelings. At least, it is promising to investigate some of the regular families of graphs which tessellate the plane.

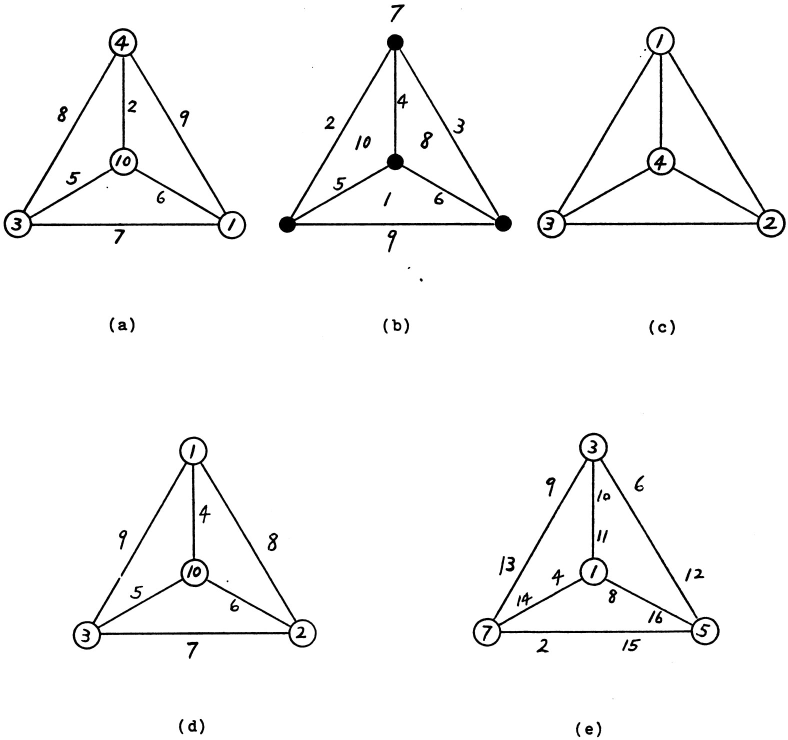
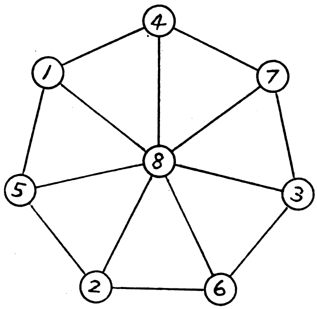


Figure 1. (a) Magic of type (1,1,0).
(b) Magic of type (0,1,1).
(c) Consecutive of type (1,0,0).
(d) Consecutive of type (1,1,0).
(e) Magic of type (1,2,0).



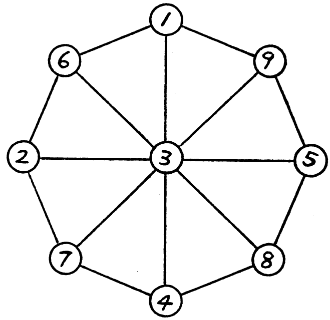
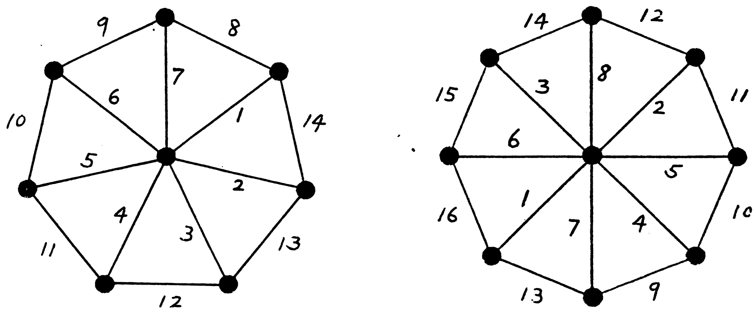
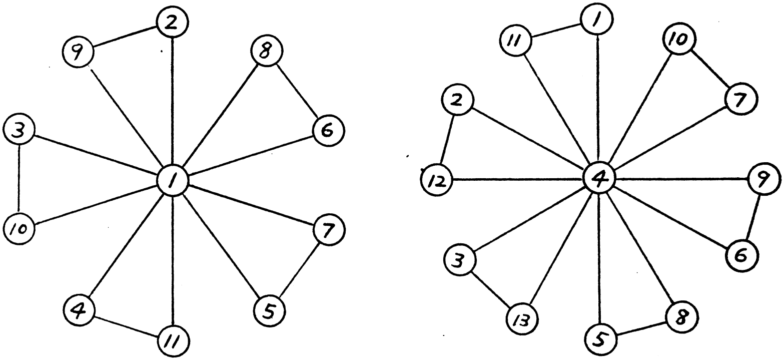


Figure 2. (a) W_7 under L_1.
(b) W_8 under L_2.



(a) (b)

Figure 3. (a) W_7 under L_3.
(b) W_8 under L_4.



(a)

(b)

Figure 4. (a) F_5 under L_5.
(b) F_6 under L_6.

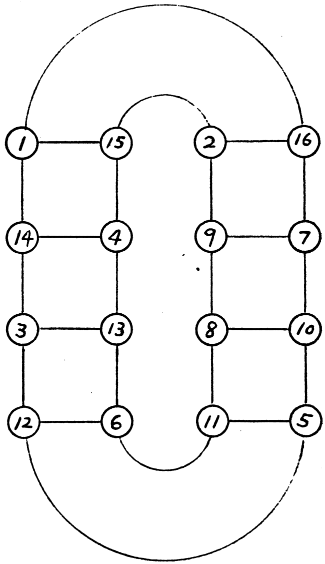
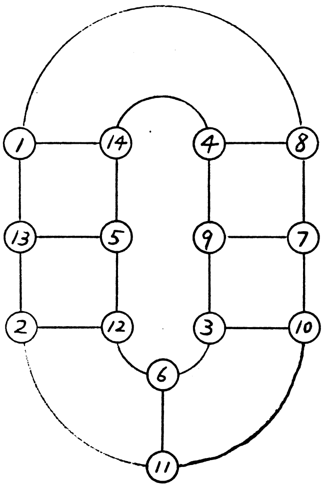
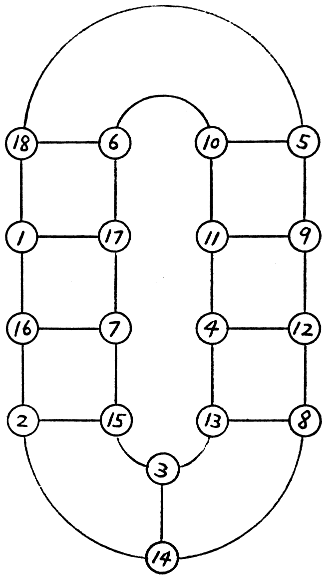


Figure 5. R_8 under L_7.

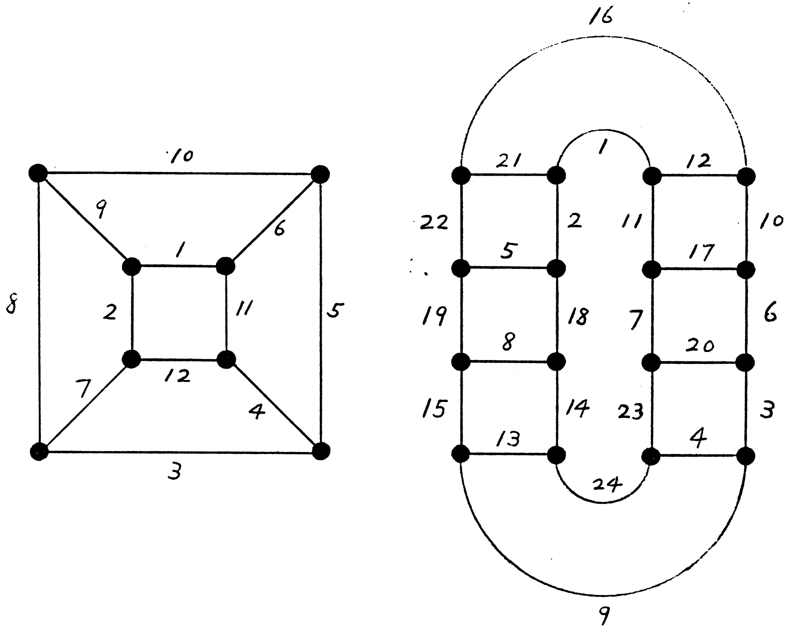


(a)



(b)

Figure 6. (a) R_7 under L_8.
(b) R_9 under L_9.



(a) (b)

Figure 7. (a) R_4 under L_{10}.
(b) R_8 under L_{10}.

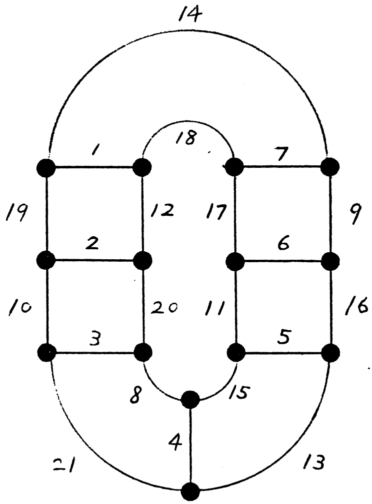
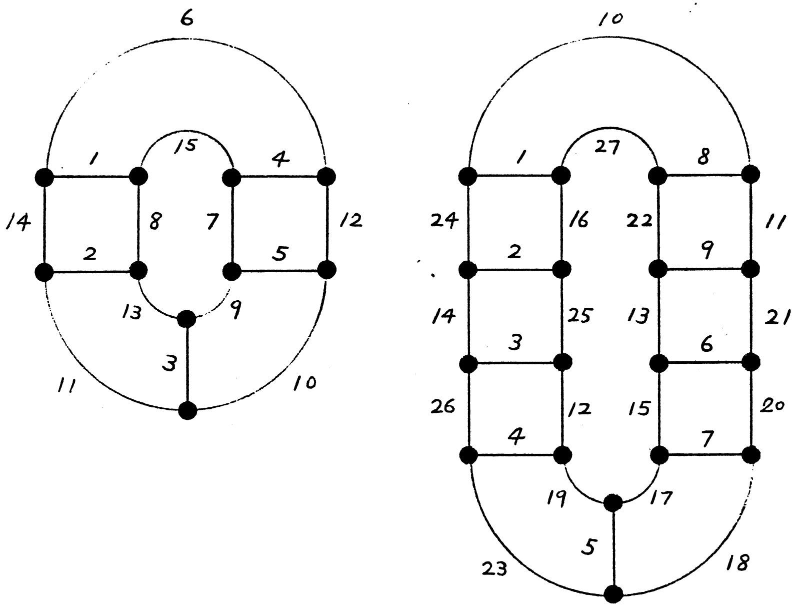
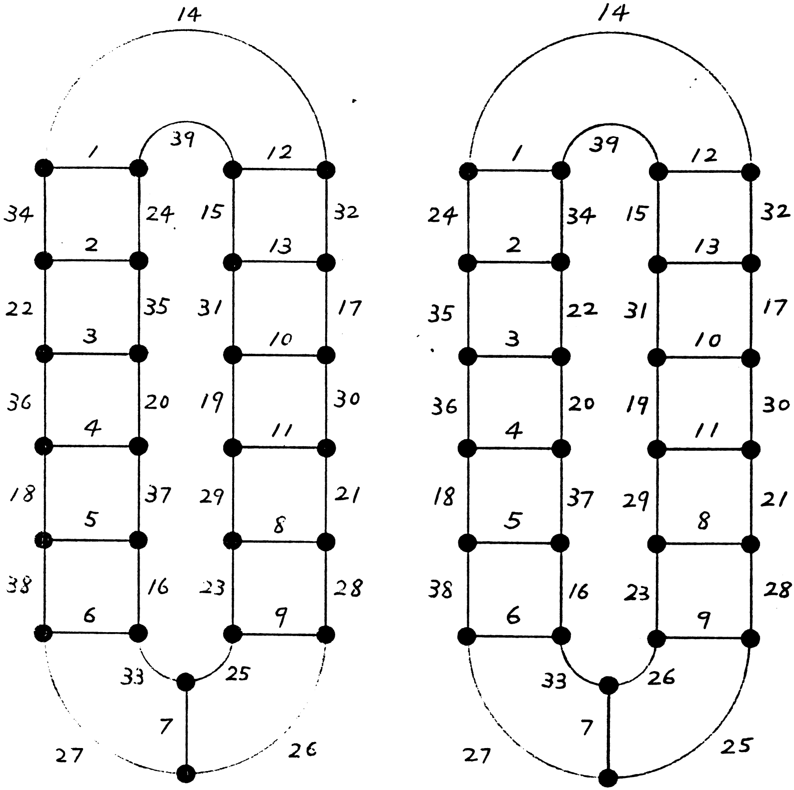


Figure 8. R_7 under L_{11}.



(a) (b)

Figure 9. (a) R_5 under L_{13}.
(b) R_9 under L_{13}.



(a) (b)

Figure 10. (a) R_{13} under L_{12}.
(b) R_{13} under L_{13}.

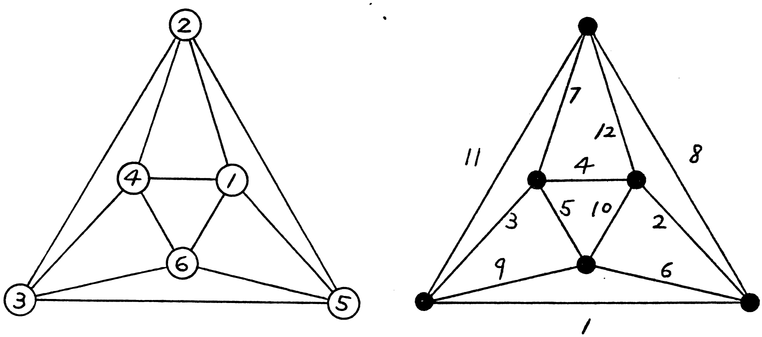


Figure 11. Complementary consecutive vertex and edge labelings of octahedron.

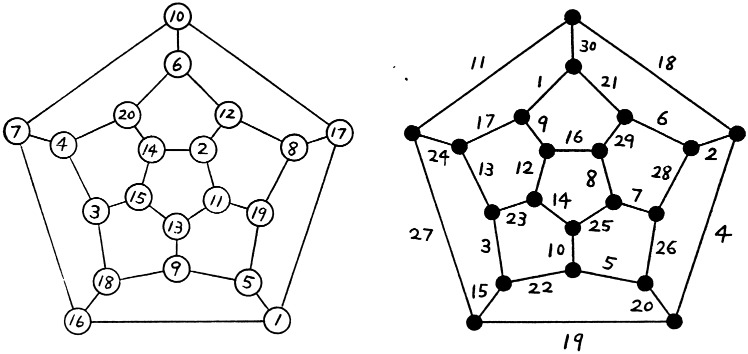


Figure 12. Complementary consecutive vertex and edge labelings of dodecahedron.

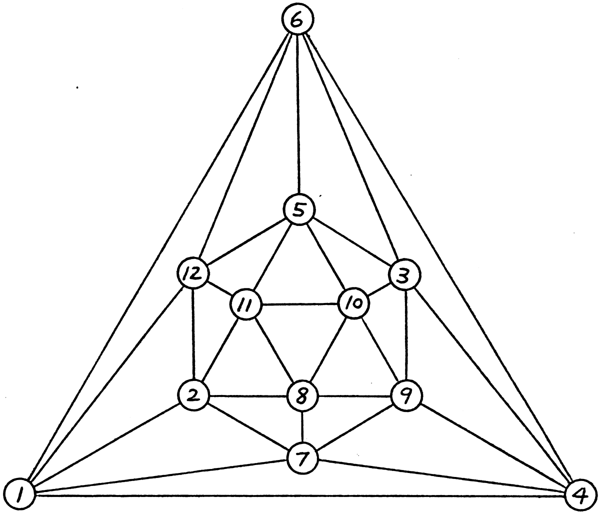


Figure 13. A consecutive vertex labeling of icosahedron.

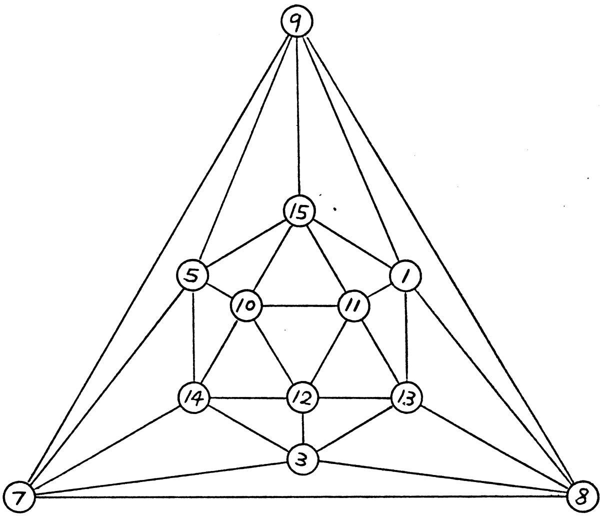


Figure 14. Vertex labels of a type (1,2,0)
magic labeling of icosahedron.

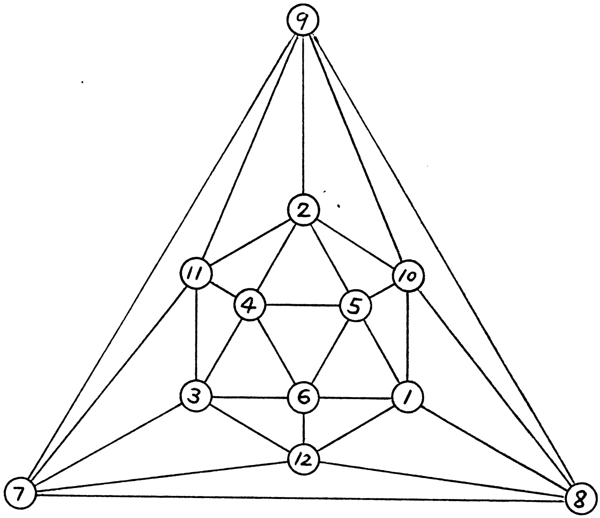


Figure 15. Vertex labels of a type (1,3,0) magic labeling of icosahedron.

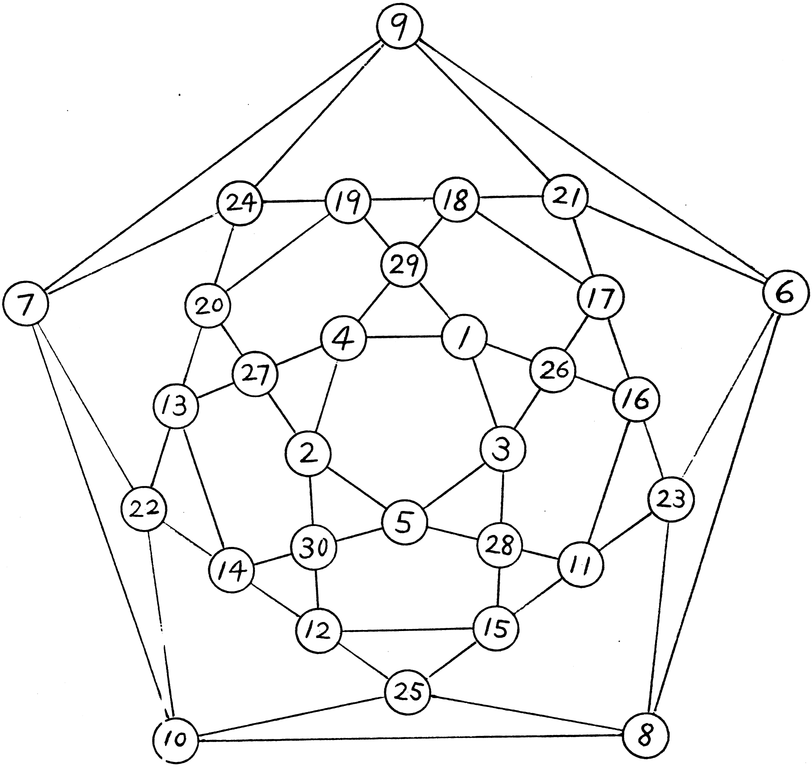


Figure 16. Vertex labels of a type (1,1,0) magic labeling of icosidodecahedron.

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